Toward A Solution to the Uncovered Interest Rate Parity Puzzle

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UNCOVERED INTEREST RATE PARITY PUZZLE

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Abstract

The concept of Uncovered Interest Rate Parity (UIP) suggests that the relationship between the percentage change in the spot rate and a one period lagged value for the interest rate differential, denoted by beta, should be plus unity. For more than 25 years economists have been baffled by the fact that the estimates of beta are typically negative. Furthermore, more recent empirical evidence suggests that beta is often positive for high inflation countries. The objective of this paper is to make progress towards solving this puzzle. We develop and simulate a general UIP model wherein risk averse agents forecast interest rate differentials. We show that a constant risk premium leads to four regimes in which the fx market can operate, and each regime generates a different solution equation for the dynamics of the spot rate. The model implies that the beta coefficient will be biased downwards, perhaps significantly, below plus unity even though the risk premium is constant. The empirical section of the paper uses our theoretical model and actual interest rate differential data for the US versus four OECD countries to calculate a time series for the change in the exchange rate. Using the calculated changes in the spot rate and lagged actual values for the interest differentials, we estimate Fama type equations with results that are consistent with the empirical literature on this subject.

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1. Introduction

The so-called forward bias puzzle has baffled economists for more than 25 years. The difference between the log of the forward exchange rate and the log of the spot rate, i.e., the forward discount or premium, should predict the future percentage change in the spot rate in a linear fashion with a coefficient, referred to as Fama’s beta, of plus unity. Equivalently, ex post uncovered interest parity, UIP, holds if the current period’s home minus foreign nominal interest rate differential predicts the next period’s percentage change in the spot rate, again in a linear fashion with a coefficient of plus unity, which is also called Fama’s beta coefficient. However, empirical tests of these two equations generate beta coefficients that are usually negative. Indeed, Froot and Thaler (1990) report that the average value for beta in more than 70 empirical estimates is -0.88. This is typically called the forward bias puzzle but it also represents a UIP puzzle.1 More recent studies confirm that the puzzle still exists2.

Burnside (2008) points out that the UIP puzzle is well documented for advanced economies with relatively low inflation. However, there is some noteworthy evidence to the contrary. The so-called “extreme support” literature finds some evidence consistent with UIP among OECD countries when Fama regressions use only outlier values for interest rate differentials.3 Burnside, Eichenbaum, and Rebelo (2007) show that estimates of $\beta$ are positively correlated with the inflation rates in 16 OECD countries. Consistent with this is the fact that Baillie and Kilic (2006), Chinn (2006), Huismann, et al (1998), and Gourinchas and Tornell (2004) all get positive estimates of beta for Italy. With regard to emerging nations (who are likely to have relatively high inflation rates) scholars consistently obtain evidence in support of UIP.4

Many explanations to the puzzle have been proposed. The work of Fama (1984) suggests that there might be a missing variable, denoted here by X, in the estimated equations. If so then estimates of the beta coefficient will be biased away from plus unity if there is a nonzero covariance between X and (i) the forward premium, or, (ii) the interest rate differential. The UIP literature has suggested two possibilities, namely that X contains a risk premium and/or a

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1 See McCallum (1994)
variable that reflects exchange rate prediction errors. Engel (1996) surveys the literature dealing with a variable risk premium which concludes that variations in the risk premium will not be large enough to generate a negative beta unless unrealistically large values for the coefficient of risk aversion are assumed.

On the other hand, Froot and Frankel (1989) and more recently Campbell, et al (2007) maintain that variations in the risk premium are trivial compared to prediction errors. Mark and Wu (1999) replicate the puzzle by assuming that an expectations error is negative correlated with the interest rate differential. Lewis (1995) points out that persistent expectations errors need not violate Rational Expectations if they arise via a Peso problem. However, the empirical evidence of a negative beta is so pervasive that it is difficult to accept the idea that Peso problems are this prevalent. Another noteworthy explanation (that is unrelated to the existence of a missing variable) appears in McCallum (1994). He suggests that the puzzle exists because of the existence of central bank reaction functions, which create a simultaneous equation bias when estimating Fama betas.

More recent literature has taken a new approach, namely that the UIP puzzle is tied up with the fact that fx speculation in the form of carry-trade: (a) might not occur very often, and/or (b) when it does occur, the volume of carry-trade funds is insufficient to generate \textit{ex ante} UIP. The “extreme support” literature mentioned above is one example of this new approach. It implies that there is a risk premium associated with carry-trade that is, perhaps, very large. Consistent with this idea is the literature related to the “disaster risk” concept of Barro (2006). Burnside, Eichenbaum, Kleshchelski, and Rebelo (2008), BEKR, Jurek (2008), and Farhi, Fraiberger, Gabalex, Ranciere, and Verdelhan (2009), FFGRV, find that the excess return from carry trade is in the range of 5% to 6.5%, depending on the relevant time interval. They contend that this represents a risk premium that includes “disaster risk”. Therefore, the concept of “disaster risk” offers an explanation as to why excess returns from carry-trade are so high.\footnote{Sercu and Vandebroek (2005) thoroughly investigate the statistical properties of X.}

\footnote{This literature compares the returns from unhedged versus hedged carry-trade (where the latter uses put and call options to protect against major disasters) and finds that Disaster Risk represents at least 25% of the total risk premium.}

\footnote{Farhi and Gabaix (2008) assume that disaster risk always declines gradually in countries with relatively high disaster risk (and, hence, high interest rates). This generates a negative beta coefficient. However, BEKR (2008) find a zero covariance between excess returns from carry trade and a host of risk measures.}
Bacchetta and van Wincoop (2008) assume that agents must incur large fixed costs in order to be able to adjust their portfolios in every period. Thus, they adjust relatively infrequently. Brunnermeier and Pedersen (2008) and Brunnermeier, Nagel, and Pedersen (2008), BNP, believe that the fx market often faces liquidity constraints. In all cases where market imperfections exist, there are insufficient speculative funds within any one time period to completely eliminate potential profits from carry-trade. Thus, funds move toward a high interest rate country for many consecutive periods, thereby steadily appreciating its currency and generating a negative beta coefficient.

In sum, the more recent literature suggests that UIP does not hold because the frequency and/or volume of carry-trade is insufficient to generate UIP. This, in turn, implies the existence of (a) a large risk premium that probably includes “disaster risk”; and/or (b) significant costs associated with monitoring portfolios; and/or (c) market frictions that often create a shortage of liquidity in the fx market. Our paper is in the same spirit as this literature in that it suggests that an essential ingredient in the solution to the UIP puzzle is that fx speculation occurs relatively infrequently.

In an important paper, Gourinchas and Tornell (2004) incorporate interest rate expectations into a UIP model, and assume that they do so with an infinite time horizon. In their model the beta coefficient is biased downward (and can conceivably become negative) if Rational Expectations are violated when agents predict interest rate differentials.

A key technical innovation in our paper is that we add a threshold variable or “risk premium” to a version of the Gourinchas and Tornell (2004) model that assumes Rational Expectations with regard to forecasts of interest rate differentials. The existence of a “risk premium” implies that there will be an active and an inactive zone in the fx market, where it is optimal for agents to have, respectively, a nonzero fx position or a zero fx position. This drives most of the conclusions reached in our paper. The existence of an active (A) and an inactive (I) zone implies that in any two consecutive periods there are four possibilities, namely the fx market moves from A to A, A to I, I to I, or I to A. Each of these Regimes generates a different solution equation for the dynamics of the spot rate. The value for the beta coefficient and the terms in the missing X vector differ in each of the four equations.

Therefore, estimates of Fama’s beta coefficient represent a weighted average of four different betas. We show theoretically and via simulations that estimates of Fama’s beta will be
biased downward significantly if the fx market spends relatively much time in the inactive zone, even if beta equals +1 when carry-trade takes place regularly. This represents an important finding, because it means that a large constant risk premium helps to explain the UIP puzzle; a variable risk premium is not needed. This conclusion is consistent with the “extreme support” literature, because it shows how and why it is possible to have a positive value for Fama’s beta when interest differentials are large enough to induce carry-trade, but a negative beta when all values for the differentials are used.

Also, the model suggests that estimates of the beta coefficient in a Fama regression will be unstable, depending on the relative amount of time that the fx market spends in each regime. Our simulations confirm this. Finally, the paper uncovers a new missing variable that reflects agents’ current expectations about the value for the spot rate at the end of their time horizon for forecasting interest rate differentials. Simulations show that the sign of the covariance between this new missing variable and the interest rate differential greatly influences the sign of Fama’s beta coefficient. This represents an important insight into the UIP puzzle and it suggests a hypothesis as to why estimates of beta are often positive in high inflation countries.

The empirical section of the paper uses our theoretical model and actual interest rate differential data for the US versus four OECD countries to simulate a time series for the change in the exchange rate for each country. The simulated value for the change in the exchange rate in each period is determined by the appropriate solution equation, depending on the regime in which the market is operating. Then OLS is used to estimate a Fama equation using the simulated changes in the spot rate and lagged actual values for the interest differentials. This is done for each country and for reasonable combinations of parameter values. For specific parameter values our simulation results are consistent with the empirical literature.

2. A Simple Risk Adjusted UIP Framework
2.1 Assumptions and Optimizing Behavior

The model assumes that at the beginning of period t, the home minus foreign nominal interest rate differential, ID(t), is zero, and that the log of the spot rate, s(t), in dollars per unit of fx, equals its exogenous and constant long run equilibrium value, s_o*. Then ID(t) increases exogenously in period t and fx speculation via carry-trade takes s(t) away from s_o* instantaneously. However, the spot rate is expected to return gradually to its constant long run
equilibrium value, $s_o^*$, as $\text{ID}(t)$, decays over time. That is, the model assumes that \textit{ex ante} UIP always holds at the end of any period when, initially, it is profitable to have a nonzero fx position.\footnote{Froot and Frankel (1989), Frankel and Chinn (1993) and Cavaglie, et al (1994) use survey data on expected changes in the spot rate to estimate Fama regressions. They generally get estimated values of $\beta$ that are positive, thereby lending credibility to the idea that \textit{ex ante} UIP holds.}

Also, at any time “$t$”, there is a threshold magnitude or “risk premium” $\rho(t)$ that agents compare with the anticipated one period gross profit from carry-trade.\footnote{See survey evidence reported in Cheung and Chinn (2001) suggests that the risk premium is a positive function of exchange rate volatility. Along these lines, Bollerslev and Melvin (1994) find a positive relationship between the “bid-ask” spread (an alleged proxy for risk) and exchange rate volatility. Finally, we show below that our model generates variations in the spot rate that become more volatile as the absolute value for the interest rate differential increases.} Sercu and Vandebroek (2005) point out that $\rho(t)$ might reflect transactions costs, information costs, and/or opportunity costs, as well as the traditional required payment associated with exchange rate risk. As pointed out above, more recent literature suggests that $\rho(t)$ also includes “disaster risk”. For simplicity we refer to $\rho(t)$ as simply the “risk premium” in quotation marks.

Equation (1a) assumes that the “risk premium” is a positive function of $\text{ID}(t)$ and that it never becomes smaller in absolute value than $\rho_o$. A positive relationship between $\text{ID}$ and $\rho(t)$ is consistent with the work of BNP (2008), who find that the distribution of excess returns from carry trade becomes progressively more negatively skewed as the absolute value for $\text{ID}$ increases. Thus, the probability of a huge loss (and, hence, the risk) increases with $\text{ID}$.\footnote{The sign of the “risk premium” must be the same as the sign of the interest rate differential. To explain, let the absolute value for $\rho_o$ be 3%, and $\text{ID}(t) = +0.05$. Thus, agents who take a long position in dollars will earn a risk adjusted excess return of 2%. That is, $\text{ID}(t) - \rho_o = +.05 - .03$, if $\rho_o$ is a positive number. On the other hand, if $\text{ID}(t) = - .05$ and the absolute value for $\rho_o$ remains at 3%, then $\rho_o$ must be defined as a negative number in order to get: $\text{ID}(t) - \rho_o = -.05 - (-.03) = -.02$. There is now a negative risk adjusted return from going long on dollars, but a positive return of 2% from going long on fx.} For simplicity, the Simple UIP model holds the absolute value for $\rho(t)$ constant at $\rho_o$, i.e., $\psi = 0$. Later we allow it to vary endogenously.

$$\rho(t) = \rho_o + \psi \text{ID}(t) \quad \psi \geq 0$$  \hspace{1cm} (1a)

It is a well known stylized fact that interest rate differentials are highly persistent, but that they decay eventually. Therefore, we follow Gourinchas and Tornell (2004) as well as Bacchetta and van Wincoop (2008) by assuming that agents anticipate an average rate of decay per period,
\( \phi^*(t) \), for the known value of the current period’s interest rate differential, ID(t).\(^{11}\) They can revise this expected rate of decay every period, i.e., \( \phi^*(t+1) \) need not equal \( \phi^*(t) \). Also, following Bacchetta and van Wincoop, the forward looking time horizon for predicting the future value for any nonzero ID(t) is finite and denoted by \( k \), which is determined endogenously as follows.\(^{12}\)

At time “t” agents use the exogenously given value for \( \rho(t) \) to decide if it is optimal for them to take a nonzero fx position in period t by engaging in carry-trade.\(^{13}\) If this is desirable, then (in period t) they also decide on the optimal time horizon for their speculative position, denoted by \( n(t) \); that is, should they plan to have a nonzero fx position for only one period, or for \( n > 1 \) periods? Appendix A shows that the rigorous decision making process requires agents (at time t) to formulate expectations about the values for ID and changes in the spot rate for each future period from t until t+\( k \). The appendix also proves that in our Simple UIP framework the same results are obtained via a simple Rule, namely that carry-trade is expected to be profitable if \( ID(t) \) exceeds \( \rho(t) \).\(^{14}\)

To explain, agents first compare the absolute values of ID(t) and \( \rho(t) \). If the former is less than the latter, then it is optimal to have a zero fx position in period t. This is referred to as the inactive zone; in this case \( n(t) = 0 \). However, if ID(t) exceeds \( \rho(t) \) in absolute value, then a nonzero position in fx is optimal in period t. Agents then compare what they expect will be the value for ID in the next period, \([1-\phi^*(t+1)]ID(t)\), with \( \rho(t+1) \), and decide (in period t) if they believe that a nonzero fx position will be profitable for two periods. If so, then they repeat the calculation for each future period, until they believe that ID(t) will have decayed enough that their expectation about the value for \( ID(t+k) \) is less than or equal to \( \rho(t+k) \). When k becomes this large, then agents no longer attempt to predict ID any further out; that is, they attempt to forecast ID for only k periods. Thus, \( n(t) \), the optimal time horizon for their nonzero speculative position in period t, equals \( k-1 \).

\(^{11}\) They need not believe that the rate of decay will be constant over time. Rather, \( \phi^*(t) \) is the constant rate of decay that generates the same n period sum of anticipated IDs as the perhaps nonconstant \( \phi^*s \) that might exist.

\(^{12}\) The time horizon for predicting ID is set exogenously in Bachetta and Van Wincoop. Gourinchas and Tornell have an infinite time horizon.

\(^{13}\) For ease of exposition, the discussion usually relates only to the case where ID(t) is positive.

\(^{14}\) Bacchetta and van Wincoop (2008, p. 20) point out that: “Many large financial institutions do not bother to try to outperform the random walk when forming expectations of the exchange rate one month or more into the future.” This is consistent with the findings of Sarno and Valente (2006).
This simple Rule for deciding if it is desirable to engage in carry-trade, and (if so) the number of periods, \( n \), over which a nonzero fx position is optimal is represented by condition (1b). When the absolute value for the “risk premium” is fixed exogenously (\( \Psi = 0 \)), then the \( \rho_o \) term on the rhs of (1b) is the “risk premium”. However, if the “risk premium” varies endogenously (\( \Psi > 0 \)), then the \( \rho_o \) term on the rhs is the constant term in (1b). As stated above, when condition (1b) is satisfied, then \( k \) is increased by one unit at a time until (1b) is violated, in which case \( n(t) = k - 1 \).

\[
\left| [1 - \phi^*(t)]^{k-1} \cdot ID(t) \cdot (1 - \Psi) \right| > \left| \rho_0 \right| \quad \ldots \ldots k = 1 \ldots n(t) + 1 \quad (1b)
\]

For any given \( ID(t) \), \( \phi^*(t) \) and \( n(t) \), agents calculate the anticipated cumulative excess interest on dollar assets from carry-trade in period \( t \) according to equation (2a). \( F[ \ ] \times ID(t) \) is the sum of the current and expected, over the time horizon \( n(t) \), values for the interest differential. Equation (2a) implies that if \( ID(t) > 0 \) and a nonzero speculative position is optimal for three periods, i.e., \( n(t) = 3 \), then agents anticipate net interest from carry-trade (that involves going long on dollars and short on fx) for three periods, including period \( t \). For notational ease, \( F(t) \) refers to the value for \( F[\phi^*(t), n(t)] \), and \( F(t+1) \) is the value for \( F[\phi^*(t+1), n(t+1)] \). Note that by assumption \( F(t) = 0 \) if \( n(t) = 0 \), i.e., if it is optimal to have a zero fx position in period \( t \). Also, (2a) implies that \( F(t) \geq 1 \) if \( n(t) > 0 \). More properties of the \( F \) function are given by (2b) and (2c).\(^\text{15}\)

\[
F[\phi^*(t), n(t)] \times ID(t) = ID(t) \{1 + [1 - \phi^*(t)]^1 + \ldots [1 - \phi^*(t)]^{n(t)-1}\} \quad (2a)
\]

\[
F_1 < 0, F_2 > 0 \quad (2b)
\]

\[
F[\phi^*(t), \infty] = 1/\phi^*(t) \quad (2c)
\]

Next, define \( \Omega(t) \) in (3a) as the cumulative anticipated (as of period \( t \)) risk adjusted excess interest on dollar assets over the time horizon \( n(t) \). Each term in \( \Omega(t) \) represents one period’s expected excess interest on dollar assets (the anticipated ID for that period) minus the

\(^{15}\) (2b) indicates that: (i) \( F[ \ ] \) decreases as \( \phi^*(t) \) rises, because, for any given \( n(t) \), a larger anticipated rate of decay in ID means that the anticipated cumulative excess interest on dollar assets declines; and (ii) \( F[ \ ] \) increases with \( n(t) \), because in this case agents expect to earn excess interest on dollar assets for a longer period of time. If agents have an infinite time horizon when predicting ID, \( n = \infty \), then (2c) indicates that \( F[ \ ] = 1/\phi^*(t) \). This is the case in the Gourinchas and Tornell (2004) model.
required “risk premium”. In the case of our Simple model, where the “risk premium” is fixed exogenously, Ω(t) is given by (3b). Note that by assumption Ω(t) = 0 if a zero fx position is optimal in period t.

\[
\Omega(t) = \sum_{k=0}^{n-1} \{ID(t) \left[1-\phi^*(t)\right]^k - \rho(t+k) \}
\]

(3a)

\[
\Omega(t) = F[t]ID(t) - n(t) \rho_o(t)
\]

(3b)

### 2.2 Ex Ante UIP

Risk adjusted *ex ante* UIP exists at the end of period t if agents expect the dollar to depreciate over the next “n” periods by the anticipated cumulative risk adjusted excess interest from carry-trade in period t, i.e., by Ω(t). This will be true if the current value for the spot rate, s(t), differs from what agents believe (in period t) will be its value “n” periods later by Ω(t). In our simple version of the model the latter equals the exogenously given long run equilibrium value for the spot rate, s_o*. Therefore, the spot rate always adjusts quickly so that s(t) differs from s_o* by Ω(t), as in (4a), which assumes that the “risk premium” is fixed exogenously.

\[
s_o^* - s(t) = \Omega(t) = F[t]ID(t) + n(t) \rho_o(t)
\]

(4a)

As a numerical example, assume ID(t) = 0.08, \(\rho_o = 0.04\) and \(\phi^*(t) = 0.20\), and that initially the spot rate equals s_o*. In this case agents will expect that a nonzero fx position (going short on fx and long on dollar assets) will be profitable for a total of four time periods from t through t+3; that is, n(t)=4. This is true because \((1-\phi^*)^3 ID(t) = 0.0409 > \rho_o\), but \((1-\phi^*)^4 ID(t) = 0.032 < \rho_o\). Consequently, from (2a), \(F(t) = 1 + (1-\phi^*) + (1-\phi^*)^2 + (1-\phi^*)^3 = 2.952\), and from (3a), \(\Omega(t) = 2.952ID(t) - 4\rho_0 = 0.076\). If s* is unchanged, then ex ante UIP will hold when the dollar appreciates (instantaneously) by 7.6%.

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16 BEKR (2006) find that *marginal* excess returns from carry-trade are approximately zero, even though *average* returns are relatively high.

17 Cheung and Chinn (2001) report that fx market dealers say that over 70% of the adjustment in the spot rate to an unexpected shock to interest rates occurs within one minute.
Equation (4a) can be solved for $s(t)$ as in (4b). The term $F(t) \geq 1$ represents a type of multiplier that generates variations in the spot rate that are typically more volatile than interest rate differentials, which is consistent with a well known stylized fact. Also, since $F(t)$ increases with $n(t)$, and the latter is a positive function of the absolute value for $ID(t)$, the model suggests that, ceteris paribus, exchange rate volatility will be greater when interest rate differentials are larger in absolute value. Expression (4b) is similar to the general equation for the spot rate in Mussa (1982) and Hai, Mark, and Wu (1997) in that $s(t)$ depends on its expected long run value and on transitory but persistent deviations from this.

Rewriting (4b) one period forward gives (4c). If a zero fx position is optimal in period $t$, i.e., $\Omega(t) = 0$, then (4b) reduces to $s(t) = s_o^*$ as in (4b'). Similarly, if a zero fx position is desired in period $t+1$, i.e., $\Omega(t+1) = 0$, then (4c) reduces to (4c'). The dynamics of the spot rate are determined by equations (4b) or (4b'), and (4c) or (4c'). The solution equations use the definition of the actual rate of decay in the spot rate between period $t$ and period $t+1$, $\phi(t+1)$, implied by (4d).

\[
\begin{align*}
    s(t) &= -\Omega(t) + s_o^* = -F[t] \text{ ID}(t) + n(t) \rho_o(t) + s_o^* \\
    s(t) &= s_o^* \\
    s(t+1) &= -\Omega(t+1) + s_o^* = -F[t+1] \text{ ID}(t+1) + n(t+1)\rho_o(t+1) + s_o^* \\
    s(t+1) &= s_o^* \\
    ID(t+1) &= [1 - \phi(t+1)] \text{ ID}(t)
\end{align*}
\]

2.3 Solutions

This section solves the simple version of our UIP model wherein $\rho$ and $s^*$ are exogenous and constant. Rational Expectations is assumed to hold with regard to ID predictions. Thus, the anticipated rate of decay in ID, $\phi^*(t)$, always equals the true expected value of the actual rate of decay in ID, denoted here by $\Phi$; that is, $\phi^* \equiv \Phi$. The solution equations for the change in the exchange rate, $ds(t+1)$, appear in Table 1 as (5a), (6a), (7a) and (8a). They are

\[18\text{ Equations (4b) and (4c) are similar to equation 9 in Gourinchas and Tornell, if } \rho = 0, \text{ and } n = \infty, \text{ i.e., } F = 1/\phi^*(t) \text{ from (2c).}]}
obtained by subtracting the appropriate expression for \( s(t) \), either (4b) or (4b’), from the appropriate expression for \( s(t+1) \), either (4c) or (4c’).\(^{19}\)

The four possible Regimes are: **Regime 1**, where a **nonzero** fx position is optimal for two consecutive periods, \( t \) and \( t+1 \); **Regime 2**, where a **nonzero** fx position is desired in period \( t \), but not in period \( t+1 \); **Regime 3**, where it is optimal to have a **zero** fx position for two consecutive periods, \( t \) and \( t+1 \); and **Regime 4**, where agents desire to have a **zero** fx position in period \( t \), but a **nonzero** position in period \( t+1 \). The four solution equations can be rewritten in the form of a typical Fama equation by using (4d) to substitute for \( d\text{ID}(t+1) \) and \( \text{ID}(t+1) \). This gives (5b), (6b), (7b), and (8b), with the theoretical value for \( \beta \) and the terms in \( X \) within each Regime given by (5c) and (5d) through (8c) and (8d).\(^{20}\)

Note well that the theoretical value for \( \beta \) differs in each Regime. The \( \beta 1(t) \) term in (5c) indicates how \( \text{ID}(t) \) affects \( ds(t+1) \) when it is optimal to have a nonzero fx position for at least two consecutive periods. \( \beta 1 \) corresponds to the beta in traditional thinking about UIP. Appendix C proves that if agents correctly predict the rate of decay in ID, then \( \beta 1(t) = 1 \). Also, (6c) indicates that \( \beta 2(t) = F(t) \), and we know from the discussion of equation (2a) that \( F(t) \geq 1 \).\(^ {21}\) Next, (7c) indicates that \( \beta 3(t) = 0 \), because changes in the spot rate are not directly affected by interest rates when IDs are too small in absolute value to make nonzero fx positions desirable.

Finally, (8c) shows that \( \beta 4(t) = -F(t+1)[1-\phi(t+1)] \), where \( \phi(t+1) \) is the most recent actual rate of decay in ID. In Regime 4 agents go from a zero fx position in period “\( t \)” to a nonzero position in “\( t+1 \)” This means that the absolute value for ID in period \( t \) is less than or equal to the absolute value for the risk premium; but the absolute value for ID in period \( t+1 \) exceeds that for the risk premium. This, in turn, implies that \( \phi(t+1) < 0 \) in Regime 4, i.e., ID(t) does not decay between “\( t \)” and “\( t+1 \)” but rather it grows in absolute value. Therefore, \([1-\phi(t+1)] > 1\), and, since \( F(t+1) \geq 1 \), it follows that \( \beta 4(t) < -1 \).\(^{22}\) In sum, the simple UIP model generates the

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\(^{19}\) Appendix B contains an outline of the derivations.

\(^{20}\) It can be proven that Conventional Thinking about UIP as well as the model in Gourinchas and Tornell (2004) are nested within our general framework. Proof of this is available upon request.

\(^{21}\) The logic here is that Regime 2 represents the case where the market moves from an active to an inactive zone. Thus, agents dump all dollars that they had been holding, thereby driving the dollar down, \( ds(t+1) > 0 \), perhaps significantly.

\(^{22}\) The intuition is that when a nonzero speculative position (long on dollars) suddenly becomes profitable, the dollar instantaneously appreciates, \( ds(t+1) < 0 \), to satisfy risk adjusted \textit{ex ante} UIP; this appreciation can conceivably be substantial if \( n(t+1) \) is large.
following theoretical values for the beta coefficient: **Regime 1**: $\beta_1 = 1$; **Regime 2**: $\beta_2 \geq 1$; **Regime 3**: $\beta_3 = 0$; **Regime 4**: $\beta_4 < -1$

Estimates of Fama’s $\beta$ represent a weighted average of $\beta_1$ through $\beta_4$. Therefore, if the fx market spends relatively much time in Regime 3, it follows that a weighted average value for $\beta$ is likely to be substantially less than plus unity even though $\beta_1(t) = 1$. It is commonly believed that a downward bias for beta requires a variable risk premium. However, clearly a constant $\rho$ pushes $\beta$ downward, provided that $\rho$ is large enough that the fx market operates in the inactive zone for more than a trivial percent of the time. This represents an important insight into the UIP puzzle.

Another conclusion follows from the fact that the estimated value for $\beta$ in a Fama regression depends on the relative amount of time that the fx market spends in each regime. If this varies over time, then Fama regressions will yield unstable estimates of $\beta$. Finally, the missing X term in each Regime includes $d_{s*}$. In this simple version of the model $s_0*$ is the exogenous long run equilibrium value for the exchange rate, and it is constant. However, in a more complex version of the model below, $s*$ is endogenous, and it reflects agents’ current expectations about the value for the spot rate at the end of their time horizon for forecasting interest rate differentials. **To our knowledge, this new missing variable has not been accounted for in previous research on the UIP puzzle.**

### 3. Simulations of the Simple Model

We use quarterly 90 day (at annual rates) nominal interest rate differentials data for the US versus Germany, UK, Japan and Canada from 1979:1 to 2000:4. The plan is to use our model and the ID data to generate a time series for the change in the exchange rate for each country. We then use the ID data and the calculated values for the change in the spot rate to estimate Fama type regressions. This section first simulates the simple UIP model, wherein the absolute value for “risk premium” is exogenous and constant, i.e., $\psi = 0$ in (1a). The second

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24 We are grateful to Menzie Chinn for supplying us with the data used in Chinn and Meredith (2004) and Chinn (2006).
version allows the “risk premium” to vary positively with ID. Both versions assume that $s^*$ is fixed at $s_o^*$.

3.1 The Procedure

Initially we use the following values for the “risk premium”, $\rho_o = \pm 0.03, \pm 0.04; \pm 0.05; \text{ and } \pm 0.06$. These values seem reasonable because Baldwin (1990) suggests that speculation might not be profitable unless the absolute value for the interest rate differential is at least 3%. Also, Sarno, et al (2006) calculate Sharpe ratios and conclude that 83% of the time, IDs are too small (in absolute value) to make fx market speculation profitable. Furthermore, as pointed out above, BEKR (2008) and FFGRV (2009) find that the average excess return from carry trade (which they believe represents a risk premium) is in the range of 5% to 6.5% annually. Part (b) of Table 2 shows that when the “risk premium” is in the 3% to 6% range, then the Simple model places the fx market in Regime 3 between 33% and 90% of the time for Germany, the UK, and Japan. For Canada the numbers are between 84% and 100%.

For each country and for each value for $\rho_o$ we estimate an AR1 via (9a) in order to obtain the average rate of decay in ID, $\Phi$. The estimated values for $\Phi$ are presented in Table 2. Next we calculate the time horizon “n” over which a nonzero speculative position is optimal by using the procedure surrounding (1b), with $\psi = 0$ initially. For any one set of parameter values, this is done for each period, thereby giving a time series of values for “n” in each country. Then the constant value for $\Phi$ and the calculated time series of “n” values are used to calculate a time series of values for the exchange rate multiplier $F[ ]$ in equation (2a). Next $ds(t+1)$ is calculated for each time period using actual ID data and the appropriate solution equation from 5a, 6a, 7a and 8a, depending on which regime the market is in. Finally, the actual ID(t) data and the calculated values for $ds(t+1)$ are used to estimate (via OLS) a Fama regression, such as (9b), for each possible combination of parameter values.

$$ID_t = a + (1- \Phi)ID_{t-1} + \varepsilon_t, \text{ for } t = 1, \ldots, 86 \quad (9a)$$

$$ds(t+1) = \alpha + \beta*ID(t) + u_t \quad (9b)$$

25 As pointed out above, the risk premium has the same sign as the interest rate differential.

26 Similarly, Burnside, et al (2006) concludes that transactions costs are significantly large compared to the profit from excess return predictability.
3.2 Results

Part (a) of Table 2 gives the results of the simulations of the Simple model that assumes an exogenous and constant value for $s_0^*$ and an exogenous “risk premium”. When the “risk premium” takes on values between 3% and 6%, the estimated value for $\beta$ in each country is positive and much less than +1, ranging between 0.01 and 0.29. Therefore, the model consistently generates a significant downward bias for Fama’s $\beta$. Recall from Table 1 that in Regime 3 we have $\beta_3 = 0$. Thus, the simulation results strongly support our theoretical conclusion that the average value for $\beta$ will be substantially less than +1 if the fx market spends relatively much time in Regime 3. In sharp contrast to conventional thinking, this occurs with a constant absolute value for the risk premium.

In each country the estimated value for Fama’s $\beta$ increases in Table 2 as the risk premium decreases from 0.06 to 0.03. This occurs because Fama’s $\beta$ will approach the value for $\beta_1 = 1$ as it becomes optimal to hold a nonzero fx position in progressively more time periods. Thus, as an experiment we gradually reduce $\rho_o$ to the unrealistically small absolute value of 0.005. The estimated values for Fama’s $\beta$ grow progressively larger, just as suggested by the model. They, however, never equal +1, because Part (b) of Table 2 indicates that the fx market occasionally operates in Regime 3 even if the “risk premium” is 0.005.

Even though the existence of a risk premium (and hence an inactive zone in the fx market) pushes estimates of $\beta$ below +1, none of the estimates of $\beta$ in the Simple model are negative. Thus, we now investigate whether a variable risk premium can push $\beta$ downward sufficiently to get $\beta < 0$. Since the $\psi$ parameter in (1a) can conceivably lie between zero and +1, we report the results using values of $\psi = 0.1$ and 0.3. These are shown in the first two rows (for each country) in Table 4. This complication consistently reduces the algebraic value for estimates of Fama’s beta coefficient. When $\psi = 0.10$ (and $\rho_o$ is 3% to 6%) then estimates of Fama’s $\beta$ are reduced by 0.01 to 0.07, with an average of roughly 0.03. With the same range of values for $\rho_o$ the reduction in the estimates of $\beta$ when $\psi = 0.3$ is between 0.02 and 0.19, with a mean of approximately 0.09. However, none of the estimated $\beta$ coefficients are negative.\footnote{Experiments with $\psi = 0.50$ yielded slightly larger reductions in estimates of $\beta$, but none were negative.}
4. A More Complex UIP Framework

4.1 Theory

Recall that the work of Fama suggests that the UIP puzzle might be caused by a correlation between ID and one or more missing variables, X. One component of X in every solution equation is \( ds^* \).\(^28\) Therefore, it seems prudent to consider the possibility that ID and \( ds^* \) might be correlated. To that end we assume that the anticipated (as of period t) change in \( s^* \) for period \( t+n \), denoted by \( ds^*(t+n) \), is a linear function of the actual or anticipated interest rate differential one period earlier, \( (t+n-1) \), as in (10a). We refer to \( \gamma \), which could conceivably be positive or negative, as the expectations effect.\(^29\) Realistically, the magnitude (and even the sign) of the expectations effect might differ between Regimes, and/or with the magnitude of “n” within any given Regime. However, our model and simulations simplify by assuming that the values for all of the \( \gamma \) terms are the same.

\[
\begin{align*}
\text{ds}^*(t+n) &= \gamma E[\text{ID}(t + n - 1)] \\
\text{s}^*(t+1) &= \text{s}^* + \text{ds}^*(t+1) = \text{s}^* + \gamma \text{ID}(t)
\end{align*}
\] (10a)

If \( \gamma > 0 \), then agents believe that the currency of a high interest rate country will tend to depreciate. One possible reason for this involves a combination of the Fisher Effect and PPP. If the high home interest rate exists because the home inflation rate is relatively high, then relative PPP suggests that the home currency will depreciate. There might, of course, be many other reasons for \( \gamma > 0 \). On the other hand, if \( \gamma < 0 \) then agents expect the currency of a relatively high interest rate country to appreciate in the future, as in the UIP puzzle.\(^30\)

This complication to our Simple UIP model influences whether or not it is profitable to have a nonzero speculative position. If the home interest rate is higher than the foreign rate, but \( \text{ID}(t) \) is less than the threshold magnitude, \( \rho(t) \), then no carry-trade will occur in the Simple model. However, if agents expect the home currency to appreciate when \( \text{ID}(t) \) is positive, \( \gamma < 0 \), then it is possible that the gain from \( \text{ID}(t) > 0 \) and \( \text{ds}^* < 0 \) might be sufficient to make a nonzero

\(^{28}\) In the Simple model \( \text{s}^* \) is exogenous and fixed; thus \( \text{ds}^* \) is zero in the four solution equations in Table 1.

\(^{29}\) Equation (10a) is similar to the forecasting equation in Clark and West (2006).

\(^{30}\) In the Dornbusch (1976) overshooting model, tight monetary policy at home generates \( \text{ID}(t) > 0 \), and prompts agents to anticipate an appreciation of home money in the long run, i.e., \( \text{ds}^* < 0 \).
speculative position optimal. On the other hand, when \( \gamma > 0 \) then a positive \( \text{ID}(t) > \rho(t) \) does not ensure that agents will go long on dollars, as in the Simple model. If they believe that the home currency is going to depreciate sufficiently when \( \text{ID}(t) > 0 \), then carry-trade might not be desirable. Appendix A shows that the existence of the *expectations effect* changes the condition determining whether or not carry-trade is optimal from (1b) to (11). Agents increase \( k \) each time that (11) is satisfied; when \( k \) is large enough that condition (11) is not satisfied, then \( n = k - 1 \).

\[
\left| \left[1 - \phi^*(t)\right]^{k-1} \text{ID}(t) \left(1 - \psi - \gamma\right) \right| > \left| \rho_0 \right| \quad \ldots \quad k = 1 \ldots n(t) + 1 \quad (11)
\]

Even though the existence of an *expectations effect* alters the decision to engage in carry-trade and the optimum value for “\( n \)” in each period, the solution to the model proceeds as before. That is, once it has been determined that \( n(t) > 0 \), agents borrow fx and buy dollars, thereby appreciating the dollar until risk-adjusted ex ante UIP holds, as in equation (4a). There is, however, an important difference here. The existence of an *expectations effect* means that the constant long run equilibrium value for the exchange rate, so* in (4a), is replaced by s*(t+n). As explained above, the latter represents what agents expect (as of time \( t \)) will be the value for the spot rate at the end of their speculative time horizon.

The dynamics of the exchange rate in this version of the model are again determined by subtracting (4b) or (4b’) from (4c) or (4c’), depending on the Regime in which the fx market is operating. As before, this yields a different solution equation and a different value for \( \beta \) in each Regime. These are given in Table 3, which uses (10a) to substitute for the ds* terms and incorporates the resulting expressions into the solution value for each \( \beta_i(t) \). Of particular interest is the fact that \( \beta_3(t) = \gamma \). If the *expectations effect* is negative, then the dollar will, in general, tend to appreciate when \( \text{ID}(t) > 0 \) even if no carry-trade takes place. This, in turn, implies that *the existence of a substantial “risk premium” (which places the market in Regime 3 relatively often) biases estimates of Fama’s \( \beta \) downward from \( \beta_1(t) \) toward \( \gamma < 0 \), and not toward zero as in the Simple model.*

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31 Derivations are available upon request.

32 In Regime 3 the spot rate is apt to vary randomly around s*, i.e., \( s(t) = s^*(t) + \epsilon(t) \), where \( \epsilon(t) \) is a normally distributed random term with zero mean. Similarly, \( s(t+1) = s^*(t+1) + \epsilon(t+1) \). Thus, \( E[ds(t+1)] = ds^*(t+1) + E[d\epsilon(t+1)] = \gamma \text{ID}(t) + 0 \), even though carry-trade does not occur.
Also, of interest in Table 3 is the solution value for beta when the market is in Regime 1, namely: \( \beta_1(t) = [1 - \psi + \gamma(1-\phi^\ast)^{n-1}] \). Clearly, \( \beta_1(t) = 1 \) if the “risk premium” is exogenous, \( \psi = 0 \), and if: (a) the expectations effect, \( \gamma \), is zero, or (b) the time horizon over which a nonzero fx speculative position is optimal approaches infinity, as would be true if the risk premium were zero.\(^{33}\) Clearly, \( \beta_1 < +1 \) if \( \psi > 0 \) and/or \( \gamma < 0 \). The combinations of parameter values used in simulations of the complex model are chosen so that \( \beta_1 \) is never negative, i.e., \( 0 < \beta_1(t) < 1 \). The fact that \( \beta_1(t) \) is less than +1 means that when estimates of Fama’s \( \beta \) (which are biased downward below \( \beta_1 \)) are more likely to be negative.

Furthermore, we know that Fama’s beta will approach \( \beta_1(t) \) as the relative frequency of the market being in Regime 1 increases. Hence, the model predicts that estimates of Fama’s beta will increase algebraically and eventually become positive as the absolute value for ID gets larger, just as in the “extreme support” literature. \textit{In other words, a negative expectations effect does not ensure that estimates of Fama’s beta coefficient will be negative.} A negative value for \( \gamma \) is a necessary but not sufficient condition for Fama’s \( \beta < 0 \). We also need the fx market to be in Regime 3 frequently.

4.2 Simulations

The results of the simulations of this version of the model are shown in Table 4, parts (b) and (c). We experimented with the following values for \( \gamma = -0.5, -1.0, +0.5, \) and +1.0. Since the results with \( \gamma = -0.5 \) and -1.0 were qualitatively similar, we report only the latter. When the parameter values are: \( \gamma = -1; \psi = 0, +0.1, \) or +0.3; and \( \rho_0 \in \{\pm0.03, \pm0.04, \pm0.05, \) or \( \pm0.06\} \) then \textit{the estimates of Fama’s \( \beta \) are negative in 48 out of 48 cases!} These range from a maximum of -0.13 to a minimum of -1.12, with a mean value of -0.71. This is slightly smaller (in absolute value) than the -0.88 reported by Froot and Thaler (1990) from more than 70 estimates of Fama’s \( \beta \) for many countries. When \( \rho_o \) takes on values between 4% and 6%, then the mean value for 36 estimates of Fama’s \( \beta \) becomes -0.81.

In order to test the model’s prediction that estimates of \( \beta \) will be unstable (because the relative frequency for the market being in the inactive Regime 3 varies significantly) we run five year rolling regressions for each country, thereby generating 81 estimates of Fama’s \( \beta \) for each

\(^{33}\) This is the case in Gourinchas and Tornell (2004).
country. Figure 1 shows the results when the parameter values in every country are: $\phi^* = \Phi; \Psi = 0$; $\rho_0 = \pm 0.04$; and $\Gamma = -1$. The estimated values for $\beta$ vary greatly, with the largest (in absolute value) often about (three?) times the smallest estimate. These important results are obtained by holding parameter values fixed, whereas in reality there appears to be no a priori reason why the parameter values should be constant.

As pointed out above, the model predicts that estimates of Fama’s $\beta$ will approach the value for $\beta_3$ as the market spends relatively more time in Regime 3, the inactive zone. The simulations generate this conclusion. See, for example, Table 4, the first row in part (b) for Germany, where $\Gamma = -1$, and the “risk premium” is assumed to be exogenous. If the absolute value for the “risk premium” takes on the values 3%, 4%, 5% or 6%, then estimated values for $\beta$ are -0.23, -0.50, -0.74, and -0.91, respectively. The results for the other three countries are qualitatively similar.

We also know from section 4.1 that the model generates the theoretical conclusion that estimates of Fama’s $\beta$ will approach the value for $\beta_1 > 0$ as the market spends progressively more time in Regime 1. As pointed out above, the “extreme support” literature finds that estimates of Fama’s $\beta$ become less negative and are often positive as progressively larger absolute values for the interest rate differentials are used in the estimation process. Our simulations approach this in the following manner. For the given set of actual IDs for each country, our model predicts that it will become progressively more likely that $\beta > 0$ as the “risk premium” approaches zero, because this will place the fx market in Regime 1 in progressively more periods. We test this by assuming unrealistically small absolute values for $\rho_0$, that go as low as $\pm 0.005$. The other parameter values are the same as those used in Figure 1, namely: $\phi^* = \Phi; \Psi = 0$; and $\gamma = -1$. These simulation results are shown in the last three columns of Part (b) in Table 4. In all countries, the estimated value for Fama’s $\beta$ eventually becomes positive. These important results are consistent with the “extreme support” literature, and they show that the existence of a negative expectations effect does not ensure a negative value for Fama’s $\beta$.

All of the above simulation results assume that the expectations effect is negative. We turn now to the case where $\gamma > 0$, and the absolute value for the “risk premium” is realistically in the 3% to 6% range. The simulation results for each country when $\gamma = +1.0$ are shown in Table

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34 Other combinations of parameters generated qualitatively similar results.
4, Part (b). In all cases, Fama’s $\beta$ equals +1. This occurs because $\gamma = +1.0$ means that when the home country has a higher nominal interest rate, $ID > 0$, then agents expect the home currency to depreciate by the same percent as $ID$. Consequently, it never pays to have a nonzero fx position. Therefore, the market is always in Regime 3 and Fama’s $\beta = \beta_3 = \gamma^{35}$

Therefore, we tried $\gamma = +0.20$, while holding $\psi = 0$. A value for the *expectations effect* of $+0.20$ implies that agents expect home money to depreciate when the home interest rate is relatively high (perhaps via the Fisher Effect and PPP), but the expected depreciation wipes out only 20% of the gain from carry-trade. The results are shown in the second line of Part b in Table 4. The estimates of Fama’s $\beta$ range in value between $+0.20$ to $+0.45$ when the “risk premium” lies between 3% and 6%. These estimated values for $\beta$ are consistent with the fact that positive estimates for $\beta$ in emerging countries are often substantially less than +1.

5. Conjectures

This paper makes progress toward solving the UIP puzzle, but a complete solution requires a rigorous theory with regard to the *expectations effect*, i.e., what determines the sign of $\gamma$ in equation (10a). A rigorous investigation of this is beyond the scope of this paper, but this section makes several conjectures along these lines, some of which have been alluded to above. First, recall from Section 1 that estimates of Fama’s beta coefficient are consistently negative in relatively low inflation OECD countries when all values for ID are used to estimate a Fama regression. However, estimates of Fama’s $\beta$ are algebraically larger and often positive for (a) OECD countries if only outlier values for ID are used, and (b) emerging nations, whose inflation rates are likely to be higher.

**High Inflation Countries**

When agents expect the home inflation rate to exceed inflation abroad, then (if home and foreign real rates of interest are equal) the Fisher effect implies that the home nominal interest rate will exceed the foreign rate, i.e., $ID(t) > 0$. Relative PPP implies that a higher home inflation rate will cause home money to *depreciate*, $ds > 0$. Thus, the Fisher Effect and PPP

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$^{35}$ Simulations with $\gamma = +0.50$ and $\psi = 0$ generated similar results, namely Fama’s $\beta \approx +0.5$ when the absolute value for $\rho_0$ was in the 3% to 6% range.
suggest $\gamma > 0$ for high inflation countries, or perhaps in countries whose inflation rates are typically low but are unusually high over some time interval, as in Germany in the early 1990s.\textsuperscript{36}

**Low Inflation Countries**

One possible explanation for a negative *expectations effect* in low inflation countries is that agents are aware of the UIP puzzle. Hence, it seems rational that they will expect home money to appreciate when ID is positive, i.e., $\gamma < 0$. The market inefficiency approach of Bachetta and van Wincoop (2008), BP(2008) and BNP(2008) discussed in Section 1 represents a promising avenue for explaining $\gamma < 0$. If agents know that there are often insufficient funds to generate *ex ante* UIP instantaneously when ID(t) > 0, then they will expect a gradual adjustment (the dollar appreciates slowly) in the spot rate, i.e., $\gamma < 0$.

Another possibility is that agents believe that ID(t) > 0 implies relatively tight home (US) monetary policy, as in a Taylor’s Rule equation. If this is expected to reduce the home (US) future inflation rate, then relative PPP implies that home money (the $) will *appreciate*, $d_s < 0$, in the future. This is consistent with the work of Clarida and Waldman (2007), who find that the dollar appreciates within a few minutes following news that the current US inflation rate has increased relative to inflation abroad. Therefore, Taylor’s Rule and PPP suggest that $\gamma < 0$ in countries whose central banks have creditable inflation targets.\textsuperscript{37} These conjectures need to be thoroughly explored in future work.

**6. Summary**

This paper develops a general UIP framework for risk averse agents that incorporates interest rate differential, ID, expectations which satisfy Rational Expectations. The existence of a large but *constant* absolute value for the “risk premium” means that the fx market often operates in a zone of speculative inactivity. Actual ID quarterly data for the US versus Germany, Canada, and Japan from 1979:1-2000:4 (and realistic values for the “risk premium”) suggest that

\textsuperscript{36} Baillie and Bollerslev (2000) and Chinn (2006) obtain $\beta > 0$ for Germany when the time interval for the estimation of a Fama regression includes the early 1990s.

\textsuperscript{37} Molodtsova and Papell (2008) specify Taylor’s Rule equations for i (in the US) and $i^*$ (abroad). They subtract the foreign Taylor’s equation from the US equation to obtain an expression for ID. Then they assume that $d_s(t+1)$ is a *negative* function of ID(t). When the resulting exchange rate equation is estimated for the US versus 12 other countries, it is able to generate statistically significant out of sample forecasts.
the market is in Regime 3 approximately 30% to 90% of the time. Many of our conclusions follow directly from the fact that agents often decide that a zero speculative position in fx is optimal.

More specifically, there are four Regimes in which the fx market can operate; each Regime has a different solution equation for the dynamics of the spot rate, and a different theoretical value for Fama’s $\beta$ coefficient. A Simple version of the model assumes a fixed value for: (i) $\rho_o$, the “risk premium” and (ii) $s^*$, the anticipated value for the spot rate at the end of the optimal time horizon for forecasting IDs. This simple model generates the following theoretical values for beta: Regime 1: $\beta_1 = 1$; Regime 2: $\beta_2 \geq 1$; Regime 3: $\beta_3 = 0$; and Regime 4: $\beta_4 < -1$.

Estimates of Fama’s $\beta$ coefficient represent a weighted average of these four values for beta, implying that such estimates can be very unstable; this is supported by our simulations. The $\beta_1$ coefficient corresponds to the value for beta in traditional thinking about UIP. In our model it represents the value for beta when carry-trade is profitable for at least two consecutive periods. The estimated value for beta in a Fama regression will be biased downward progressively more toward $\beta_3$ as the fx market spends relatively more time in Regime 3, where agents have a zero fx position. Simulations of this version of the model confirm this theoretical conclusion, but (as predicted in the Simple model) Fama’s $\beta$ never becomes negative, even when the risk premium is assumed to vary positively with ID.

Our UIP framework suggests that $s^*$ represents a new missing variable in Fama regressions. A more complex version of the model assumes that $s^*$ is correlated with ID via an expectations effect, represented by $\gamma$, that we conjecture could be positive or negative. In this version of the model, $\beta_3 = \gamma$. When the expectations effect is negative, the estimates of Fama’s $\beta$ coefficient are biased downward toward $\gamma < 0$. For various combinations of parameter values the simulations yield a negative value for Fama’s $\beta$ in 48 out of 48 cases.

Any solution to the UIP puzzle should also be able to explain the fact that scholars often obtain positive estimates for Fama’s $\beta$: (a) in advanced countries if only outlier values for ID are used; and (b) in emerging nations with high inflation rates. Our model and simulations show that there are two reasons why estimates of Fama’s $\beta$ might be positive. First, the expectations effect might be positive, as, for example, if agents believe that the currency of a high inflation country generally depreciates. Second, Fama’s $\beta$ can be positive (even though the expectations effect is
negative) if parameter values along with actual values for ID are such that carry-trade is always desirable, as in the “extreme support” literature. Our simulations are consistent with these theoretical conclusions.
References


Table 1: Solutions for the Simple UIP Model

**Regime 1: A nonzero fx position in “t” & in “t+1”**

\[ ds(t+1) = -F(t)ID(t+1) - dF(t+1)ID(t+1) + dn(t+1)\rho_o(t+1) + n(t)d\rho_o(t+1) + d s_o^* \]  
(5a)

\[ ds(t+1) = \beta_1(t) ID(t) + X1(t+1) \]  
(5b)

\[ \beta_1(t) = \phi(t+1)F(t) - dF(t+1)[1 - \phi(t+1)] \]  
(5c)

\[ X1(t+1) = dn(t+1)\rho_o(t+1) + n(t)d\rho_o(t+1) + d s_o^* \]  
(5d)

**Regime 2: A nonzero fx position in “t” but zero in “t+1”**

\[ ds(t+1) = F(t)ID(t) - n(t)\rho_o(t) + d s_o^* \]  
(6a)

\[ ds(t+1) = \beta_2(t) ID(t) + X2(t+1) \]  
(6b)

\[ \beta_2(t) = F(t) \]  
(6c)

\[ X2(t+1) = -n(t)\rho_o(t) + d s_o^* \]  
(6d)

**Regime 3: A zero fx position in “t” and “t+1”**

\[ ds(t+1) = d s_o^* \]  
(7a)

\[ ds(t+1) = \beta_3(t) ID(t) + X3(t+1) \]  
(7b)

\[ \beta_3(t) = 0 \]  
(7c)

\[ X3(t+1) = d s_o^* \]  
(7d)

**Regime 4: A zero fx position in “t” but nonzero in “t+1”**

\[ ds(t+1) = -F(t+1)ID(t+1) + n(t+1)\rho_o(t+1) + d s_o^* \]  
(8a)

\[ ds(t+1) = \beta_4(t) ID(t) + X4(t+1) \]  
(8b)

\[ \beta_4(t) = -F(t+1)[1 - \phi(t+1)] \]  
(8c)

\[ X4(t+1) = n(t+1)\rho_o(t+1) + d s_o^* \]  
(8d)
Table 2. Simulations for the Simple Model: $\gamma = 0, \psi = 0$

a) Estimations of the beta coefficient

<table>
<thead>
<tr>
<th>Country</th>
<th>$\rho_0 = \pm 0.06$</th>
<th>$\rho_0 = \pm 0.05$</th>
<th>$\rho_0 = \pm 0.04$</th>
<th>$\rho_0 = \pm 0.03$</th>
<th>$\rho_0 = \pm 0.02$</th>
<th>$\rho_0 = \pm 0.01$</th>
<th>$\rho_0 = \pm 0.005$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany</td>
<td>0.05 (1.15)</td>
<td>0.09 (1.37)</td>
<td>0.17 (1.74)</td>
<td>0.29 (2.07)</td>
<td>0.46 (2.35)</td>
<td>0.68 (2.55)</td>
<td>0.81 (2.65)</td>
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<tr>
<td>UK</td>
<td>0.02 (2.30)</td>
<td>0.05 (2.30)</td>
<td>0.11 (2.53)</td>
<td>0.20 (2.72)</td>
<td>0.34 (2.93)</td>
<td>0.57 (3.20)</td>
<td>0.74 (3.30)</td>
</tr>
<tr>
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<td>-</td>
<td>-</td>
<td>0.01 (1.40)</td>
<td>0.06 (2.13)</td>
<td>0.15 (2.38)</td>
<td>0.39 (3.08)</td>
<td>0.62 (3.42)</td>
</tr>
<tr>
<td>Japan</td>
<td>0.09 (1.93)</td>
<td>0.13 (2.04)</td>
<td>0.19 (2.18)</td>
<td>0.27 (2.32)</td>
<td>0.39 (2.49)</td>
<td>0.58 (2.80)</td>
<td>0.73 (2.99)</td>
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b) The percentage ID’s in Regime 3

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<tr>
<th>Country</th>
<th>$\rho_0 = \pm 0.06$</th>
<th>$\rho_0 = \pm 0.05$</th>
<th>$\rho_0 = \pm 0.04$</th>
<th>$\rho_0 = \pm 0.03$</th>
<th>$\rho_0 = \pm 0.02$</th>
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<td>Germany</td>
<td>90%</td>
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<td>66%</td>
<td>44%</td>
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<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>UK</td>
<td>86%</td>
<td>81%</td>
<td>70%</td>
<td>59%</td>
<td>37%</td>
<td>24%</td>
<td>%12</td>
</tr>
<tr>
<td>Canada</td>
<td>100%</td>
<td>100%</td>
<td>92%</td>
<td>84%</td>
<td>62%</td>
<td>30%</td>
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</tr>
<tr>
<td>Japan</td>
<td>83%</td>
<td>61%</td>
<td>47%</td>
<td>33%</td>
<td>24%</td>
<td>10%</td>
<td>2%</td>
</tr>
</tbody>
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Table 3. Solution Values for Beta in the Complex Model: $\gamma \neq 0$.

**Regime 1:** $\beta_1(t) = [1 - \psi + \gamma(1-\phi^*)^{n-1}]$

**Regime 2:** $\beta_2(t) = [(1-\psi)F(t) - \gamma]$

**Regime 3:** $\beta_3(t) = \gamma$

**Regime 4:** $\beta_4(t) = \{- (1-\psi)F(t+1)[1-\phi(t+1)] + \gamma(1-\phi^*)^{n-1}\}$.
Table 4. Simulations for the Complex Model

<table>
<thead>
<tr>
<th>Germany (Φ = 0.16)</th>
<th>ρ₀ = ±0.06</th>
<th>ρ₀ = ±0.05</th>
<th>ρ₀ = ±0.04</th>
<th>ρ₀ = ±0.03</th>
<th>ρ₀ = ±0.02</th>
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<th>ρ₀ = ±0.005</th>
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<tr>
<td>γ = 0,ψ = 0.1</td>
<td>0.03 (1.10)</td>
<td>0.06 (1.22)</td>
<td>0.12 (1.57)</td>
<td>0.22 (1.95)</td>
<td>0.37 (2.28)</td>
<td>0.58 (2.53)</td>
<td>0.71 (2.63)</td>
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<tr>
<td>γ = 0,ψ = 0.3</td>
<td>0.00 (0.90)</td>
<td>0.02 (1.09)</td>
<td>0.04 (1.20)</td>
<td>0.10 (1.63)</td>
<td>0.22 (2.11)</td>
<td>0.40 (2.47)</td>
<td>0.52 (2.61)</td>
</tr>
<tr>
<td><em><em>b) Endogenous ds</em> and exogenous risk premium</em>*</td>
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<tr>
<td>γ = -1,ψ = 0</td>
<td>-0.91 (-11.10)</td>
<td>-0.74 (-9.58)</td>
<td>-0.50 (-6.19)</td>
<td>-0.23 (-1.96)</td>
<td>0.09 (0.51)</td>
<td>0.44 (1.82)</td>
<td>0.67 (2.29)</td>
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<tr>
<td>γ = 0.2,ψ = 0</td>
<td>0.23 (6.86)</td>
<td>0.27 (4.40)</td>
<td>0.34 (3.63)</td>
<td>0.45 (3.17)</td>
<td>0.59 (2.97)</td>
<td>0.76 (2.81)</td>
<td>0.85 (2.77)</td>
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<td><em><em>c) Endogenous ds</em> and endogenous risk premium</em>*</td>
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<tr>
<td>γ = -1,ψ = 0.1</td>
<td>-0.98 (-10.14)</td>
<td>-0.82 (-10.09)</td>
<td>-0.61 (-8.22)</td>
<td>-0.33 (-3.64)</td>
<td>-0.02 (-0.11)</td>
<td>0.34 (1.63)</td>
<td>0.57 (2.22)</td>
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<td>γ = -1,ψ = 0.3</td>
<td>-1.12 (-9.57)</td>
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<td>-0.54 (-8.2)</td>
<td>-0.21 (-2.4)</td>
<td>0.14 (0.99)</td>
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<td>γ = 1,ψ = 0.3</td>
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<tr>
<th>UK (Φ = 0.22)</th>
<th>ρ₀ = ±0.06</th>
<th>ρ₀ = ±0.05</th>
<th>ρ₀ = ±0.04</th>
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<th>ρ₀ = ±0.02</th>
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<th>ρ₀ = ±0.005</th>
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<tr>
<td>γ = 0,ψ = 0.1</td>
<td>0.01 (2.43)</td>
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<td>-</td>
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<td>0.02 (2.24)</td>
<td>0.06 (2.46)</td>
<td>0.15 (2.75)</td>
<td>0.32 (3.08)</td>
<td>0.46 (3.27)</td>
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<td><em><em>b) Endogenous ds</em> and exogenous risk premium</em>*</td>
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<tr>
<td>γ = -1,ψ = 0</td>
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<td>0.36 (2.17)</td>
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<td>γ = 0.2,ψ = 0</td>
<td>0.20 (11.65)</td>
<td>0.24 (6.58)</td>
<td>0.28 (4.78)</td>
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<td>c) Endogenous ds* and endogenous risk premium</td>
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Appendix A: The Simple Rule for Speculating

Assumptions: Initially the risk premium is constant at $\rho_0$; then it becomes endogenous according to: $\rho(t) = \rho_0 + \Psi(t)$. Initially, in the Simple model, $s^*$ is exogenous and constant at $s_0^*$; later it becomes endogenous. Initially $s(t)$ equals $s_0^*$. Finally, the spot rate adjusts instantaneously so that ex ante UIP holds immediately after it becomes profitable to engage in carry-trade with a time horizon of “$k$”.

General Rule for Determining “$k$” and “$n = k - 1$”: At time $t$, compare $A_k(t)$ with $B_k(t)$, where

1. \[ A_k(t) = \sum [(1-\phi^*)^{k-1}] - \sum \rho(t+k - 1), \text{ where } k = 1 \ldots n+1 \]
2. \[ B_k(t) = s_0^* - s(t). \]

Procedure in the Simple Model

1. Step #1: In period $t$, set $k = 1$ so that $A_1(t) = \sum [(1-\phi^*)^{k-1}] - \sum \rho(t+k - 1)$, where $k = 1 \ldots n+1$
2. Step #2: Set $k = 2$ in $A_k(t)$
   a. $A_2(t) = \sum [(1-\phi^*)^{k-1}] - \sum \rho(t+k - 1)$, where $k = 1 \ldots n+1$
   b. Initially, in Step #2, $s_0^* - s(t) = A_1(t)$ from (1b).
   c. If $A_2(t) \leq A_1(t)$, then it is not profitable to hold a speculative fx position for two time periods. Thus, $n(t) = k - 1 = 1$.
   d. Note the condition in (c) is equivalent to $ID(t)(1-\phi^*) \leq \rho(t+1)$, because the terms in $A_1(t)$ cancel the $ID(t) - \rho(t)$ terms in $A_2(t)$.
   e. If $A_2(t) > A_1(t)$, then carry-trade is profitable. This will cause the dollar to appreciate, $ds(t) < 0$, until the new value for $s_0^* - s(t)$, equals $A_1(t)$.

2. Step #3 and Beyond:
   a. Comparing $A_k(t)$ with $B_k(t)$, is identical to the simple Rule that compares $E[ID(t+k)]$ with $\rho(t+k)$, because $B_k(t) = A_{k-1}(t)$, and $A_{k-1}(t)$ contains all but the kth terms in $A_k(t)$.
   b. In general, the simple Rule is that a nonzero speculative position with a time horizon of $k$ is optimal if $(1-\Psi)ID(1-\phi^*)^k > \rho_0$. If the absolute value for the risk premium is fixed
exogenously, then $\Psi = 0$. When $k$ becomes large enough that this condition does not hold, then $n = k - 1$.

**Procedure in the More Complex Model:** Assume that $s^*(t+n) = s^*(t+n-1) + \Gamma E[\text{ID}(t+k-1)]$

1. In Step #1: $A_1(t) = \text{ID}(t) - \rho(t)$ as before.
   a. Now when $\text{ID}(t)$ changes zero to $\text{ID}(t) > 0$, this immediately alters $s^*$ from its long run equilibrium value of $s_o^*$ to $s^*(t+1) = s_o^* + \Gamma \text{ID}(t)$.
   b. Thus, $B_1(t) = s^*(t+1) - s(t) = \Gamma \text{ID}(t)$, because $s_o^* = s(t)$ initially by assumption.
   c. A speculative position is profitable in period $t$ if $A_1(t) > B_1(t)$. This condition reduces to $(1-\Psi - \Gamma)\text{ID}(t) > \rho_o$, where $\Psi > 0$ if the risk premium is endogenous.
   d. If the condition is (c) is satisfied, then agents borrow fx and buy $\$ until $s(t)$ is driven down enough that $A_1(t) = B_1(t)$.

2. In Step 2 agents compare $A_2(t)$ with $B_2(t)$.
   a. $A_2(t)$ is the same as in Step #2 of the Simple model.
   b. $B_2(t)$ does not equal $A_1(t)$ when $s^*$ is endogenous; rather $B_2(t) = A_1(t) + \Gamma \text{ID}(t)(1-\phi^*)$.
   c. Thus, a nonzero speculative position is optimal for at least two periods if: $A_2(t) > B_2(t)$.
   d. However, the $A_1(t)$ component of $B_2(t)$ cancels the $[\text{ID}(t) - \rho(t)]$ terms in $A_2(t)$, which means that the condition to speculate becomes: $(1-\phi^*)\text{ID}(t) - \rho(t+1) > \Gamma \text{ID}(t)(1-\phi^*)$, or $\text{ID}(t)(1 - \Psi - \Gamma) > \rho_o$, i.e., the simple Rule.

3. As the time horizon, $k$, increases, eventually it will not be desirable to have a nonzero speculative position for the $k$th period forward; in this case $n(t) = k - 1$ as in the Simple model.

**Appendix B: Derivation of (5a), (6a), (7a) and (8a)**

**Derivation of (5a)**
Subtract (3c) from (3d) to get:

$$ds(t+1)= - [F(t+1)\text{ID}(t+1)+n(t+1)\rho(t+1) + F(t)\text{ID}(t) - n(t)\rho(t) + ds_o^*]$$  \hspace{1cm} (B1)

Substitute for $F(t+1) = F(t) + dF(t+1)$ to get

$$ds(t+1)= -[F(t)+dF(t+1)]\text{ID}(t+1) + F(t)\text{ID}(t) + [n(t) + dn(t+1)]\rho(t+1) - n(t)\rho(t) + ds_o^*$$  \hspace{1cm} (B2)

$$ds(t+1) = -F(t)\text{dID}(t+1) + dF(t+1)\text{ID}(t+1) n(t)[\rho(t)+d\rho(t+1)] + dn(t+1)\rho(t+1) - n(t)\rho(t) + ds_o^*$$

$$ds(t+1) = -F(t)d\text{ID}(t+1) - dF(t+1)\text{ID}(t+1) + n(t)d\rho(t+1) + dn(t+1)\rho(t+1) + ds_o^*$$  \hspace{1cm} (B3)

**Derivation of (6a):** Subtract (4b) from (4c’).
Derivation of (7a): Subtract (4b’) from (4c’).

Derivation of (8a): Subtract (4b’) from (4c).

Appendix C: Proof that $\beta_1(t) = 1$

1. Assume that $\phi^*(t) = E[\phi] = \phi(t+1)$ for all “t” and denote this as $\Phi$. (C1)
   {This implies that any nonzero ID decays at a constant rate in each period, just as anticipated.}

2. Thus (5c) becomes: $\beta_1(t) = \Phi F(t) - dF(t+1)(1 - \Phi) = \Phi F(t) - [F(t+1) - F(t)](1 - \Phi)$ (C2)

3a. Also (2a) implies: $F(t) = 1 + [1 - \Phi] + [1 - \Phi]^2 + \ldots + [1 - \Phi]^{n-2} + [1 - \Phi]^{n-1}$ (C3)
   b. And, $F(t+1) = 1 + [1 - \Phi] + [1 - \Phi]^2 + \ldots + [1 - \Phi]^{n-2}$ (C4)
   {(C4) differs from (C3) because the time horizon over which agents expect speculation to be profitable decreases by one period when ID decays partially exactly as anticipated.}

4. But (C2) can be written: $\beta_1(t) = F(t) - (1 - \Phi)F(t+1)$ (C5)

5. (C3) shows that $F(t)$ in (C5) has +1 for the first term. However, $(1 - \Phi)F(t+1)$ in (C5) does not have a +1 term; thus, each of the terms in $(1 - \Phi)F(t+1)$ cancels a similar term in $F(t)$ except for the +1 term in (C3), thereby yielding $\beta_1(t) = +1$