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Abstract: Colonel Blotto games have been widely applied in a variety of contests where the players allocate resources across a number of battlefields and the winner in each battlefield is the one with more resources there. One drawback of this standard model is the assumption that players can perfectly target their efforts toward different battlefields. In many situations, players can only imperfectly target different battlefields, with an allocation affecting more than one battlefield. We develop an extension of Colonel Blotto that incorporates this type of interrelation. The equilibria here generally utilize mixed strategies that tend to be asymmetric across battlefields. That is, at least one player will have zero probability of dividing resources equally across battlefields, even if the characteristics of the battlefields are similar.

We use advertising data from the 2002 races for governor and US senate to test the application of the Blotto model to political contests. Although the Colonel Blotto model has been used extensively to model political campaigns, we find that candidates' advertising allocations across markets are far too symmetric to fit the model.

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(I) Introduction

A variety of contests exist in which the opponents compete by expending resources. In many games of this type, the players benefit by allocating resources differently from their opponents; the game is thus one of "strategic allocative mismatch."[1] Prominent examples of this are military contests and political campaigns. One approach to modeling such situations is the classic Colonel Blotto game.[2] In this model, the winner is determined by which player puts in more resources, that is "gets there first with the most."[3] This makes the payoffs very discontinuous in the actions of the players.

In the simplest version of Colonel Blotto, the players have equal resources that they must allocate to different battlefields of equal size. On any battlefield, the winner is the player with the most resources there. The goal of the players is to maximize the expected number of battlefields that they win. Extensions of this model allow for the players to have unequal resources, for battlefields to be of different sizes, and for one of the players to have an advantage on some battlefields so as to be able to win them with fewer resources (perhaps, having gotten there first, within limits, the player does not need the most).[4, 5]

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[1] See Golman and Page (2009).

[2] Borel (1921) first proposed this game, with further analysis in Borel and Ville (1938). Modern analysis began with Tukey (1949), with follow-ups by Blackett (1954, 1958) and Bellman (1969). Only recently has a general solution to the continuous game been provided (Roberson (2006)); see also Weinstein (2005).

[3] In response to a question about who would win a battle, Gen. Nathan Bedford Forest of the Confederate Army replied, "The one who gets there first with the most men." This quotation is often used in discussions of strategy in a variety of contexts.

[4] Kovenock and Roberson (2008) consider a model of two-party competition where parties attract voters by offering redistributive policies when voters have heterogeneous loyalties to the different parties. There is a formal overlap between getting votes through redistributive policies and attracting them through campaign expenditures, as analyzed here.

[5] Colonel Blotto models of advertising go back to Friedman (1958). Colonel Blotto models of elections go back at least to Sankoff and Mellos (1972). Recently, Szentes and Rosenthal (2003) study a type of all-pay

Although the Colonel Blotto game can be straightforward to specify, equilibrium behavior can be complicated. Pure strategy equilibria exist only in special cases, such as when only two battlefields exist or when one battlefield is much larger than all the others combined. In the latter case, the pure strategy equilibrium has both players spending all resources in the larger battlefield. Pure strategy equilibria also exist if one player has such an overwhelming resource advantage that it can allocate enough resources to the battlefields to win them all, regardless of the other player's allocation.

Typically, when there are more than two battlefields, only mixed strategy equilibria exist, and can be quite complex. Consider a player maximizing over n battlefields. The marginal distributions of the mixed strategy give the probability of allocating different amounts to any battlefield, while the joint distributions give the probability of allocating different n-tuples across the battlefields. Necessary and sufficient conditions on the marginal distributions for a mixed strategy equilibrium have been specified (see Roberson (2006)). Consistent with any set of marginal distributions, multiple joint distributions can exist.

Characterizing the equilibrium joint distributions has been more difficult. Examples of these go back to Borel and Ville (1938) with other possibilities shown only recently by Roberson (2006) and Weinstein (2005). The equilibrium joint distributions can vary widely in their nature. Some are asymmetric and have supports that are of lower dimension than the n-1 dimensional space of allocations. There are also equilibria with full support such as the Hex equilibrium given by Borel and Ville; see Weinstein for a discussion of this equilibrium. In an equilibrium with full support, any neighborhood in

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auction of which a Colonel Blotto model of Electoral College competition is an example. Laslier and Picard (2002) consider a model of distributive politics that reduces to a Colonel Blotto game.

the space of allocations, including any containing the point at which resources are allocated equally to all battlefields, has positive measure.

One assumption in all these variants is that players can perfectly target their resources. That is, resources can be allocated to a single battlefield with no spillover effects to other battlefields. In many contexts, however, spillovers do exist. For example, consider advertising campaigns by firms or political candidates. A battlefield might be considered to be some group defined by gender, ethnicity, age, socioeconomic status, geographic location, or preferences. In a perfect world for the players, they would be able to target expenditures in the form of ads with different messages very narrowly aimed at each demographic group, with only that group seeing the ad designed for it. While firms or candidates do try to carry out such targeting through such actions as the placement of different ads on specific radio or television programs, perfect targeting rarely is possible. Ads that are aimed at one group will often be observed by members of other groups.[6]

Fletcher and Slutsky (2009) develop a structure that allows considering such imperfect targeting in the probabilistic voting context. Their structure has two media markets where ads can be purchased. In each market, the ads are viewed by three groups of voters: those initially supporting each candidate and those indifferent between them. The partisan types are assumed to have an intensity of preference toward their preferred candidates. Ads move this intensity in favor of the candidate running the ad. Under

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[6] Golman and Page (2009) also offer extensions of the Colonel Blotto game that involve externalities across battlefields. However, their externalities are in the payoffs, with players valuing combinations of battlefields beyond simply summing the values of each battlefield. They do not consider having the outcomes on multiple battlefields depend on the same expenditures as we do here.

probabilistic voting, the support each candidate receives from a type varies continuously with its post-campaign intensity.

In this paper, we adapt this structure to replace the probabilistic voting assumption so that, as in standard Blotto, all members of each group in a market vote for the candidate the group prefers after all ads have been shown, no matter how small the group's preference for that candidate. In addition, we generalize its applicability to situations beyond the political context.[7] We consider a situation in which multiple battlefields are grouped together because they cannot be targeted separately, and the players allocate resources to each grouping. For convenience, we will call a grouping a battlefield and each component of the grouping a sub-battlefield. We will analyze allocations across two battlefields, each of which has three sub-battlefields.[8] The two battlefields overall may be of unequal importance and the different sub-battlefields can also vary in importance. One of the players may have an advantage on some sub-battlefield so can win there, even though she allocates fewer resources to that sub-battlefield than the other player. The effectiveness of allocations can differ across battlefields or even across sub-battlefields. The effectiveness function is, however, the same for both players.[9] Unlike the typical Colonel Blotto game or the model in Fletcher and Slutsky (2009), we also assume that allocations can have diminishing rather than constant marginal effectiveness.

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[7] Imperfect targeting has been analyzed in some contexts, such as development economics (see Bibi and Duclos (2007)). However, it has not been explored in the contest literature.

[8] While it would be interesting to analyze the case where there is a continuum of sub-battlefields, such an approach would no longer be in the spirit of a Blotto model. In a continuum model, the amount a player wins would vary continuously with the resources each side expends, while in a Blotto model, there are significant discontinuities in the amounts won as functions of the resources expended.

[9] This generalization has an important implication in the political context. Since the effectiveness function may differ across markets, it incorporates the possibility of ad price differences across markets. Thus, the decision variable could be either the number of ads played in the market or the total advertising expenditure.

Our results under imperfect targeting are not only consistent with the basic nature of standard Blotto, but strengthen those results. First, pure strategy equilibria are even less likely to exist with imperfect targeting. Unlike the standard results, pure strategy equilibria may not exist even when there are only two battlefields and one is larger than the other. As follows from Theorem 1 and Lemma 2 below, the neutral sub-battlefield in one battlefield must be larger than the entire other battlefield for a pure strategy equilibrium to exist. Of course, then the equilibrium has all resources expended in the larger market.

Second, consider situations where pure strategy equilibria do not exist and the players have the same amount of resources to allocate between battlefields. If this amount lies in some range determined by the amount of the advantages the players have in the non-neutral sub-battlefields, a necessary condition on equilibrium mixed strategies is that at least one player acts asymmetrically. That is, as shown in Theorem 2 below, there cannot be equilibria where the joint distributions for both players have full support with at least one player putting no probability weight on allocations in a neighborhood around equal allocation across the two battlefields.[10] This contrasts with the standard Blotto result, which has mixed strategy equilibria with realizations where candidates allocate equally across markets. Thus, for this modified Blotto model, whether a pure or mixed strategy exists, players will use very asymmetric strategies across battlefields.

Given this, it would be possible in principle to test whether a Blotto approach is consistent with behavior in any of the specific contexts to which the analysis might apply. If Blotto were appropriate, we would expect to see asymmetric allocations across the

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[10] Note that although imperfect targeting introduces an element of having more than two battlefields, an important element of just two battlefields remains. Taking into account budget balance, each player has only a one-dimensional allocation decision.

battlefields, even when they appear to be similar. Thus, an army would allocate troops asymmetrically between two battlefronts even if the sub-battlefields in them were similar in importance and in the extent of the advantage to one army over the other. Product or political advertisements would tend to be placed more heavily in one market over another, even when the demand conditions in the markets appear to be very similar. Finally, we would expect to see significant variation in educational quality across school districts, even when the districts are alike in most measurable ways.

Whether the equilibrium is in pure or mixed strategies, players will use very asymmetric strategies across battlefields. However, since the conditions for existence of pure strategy equilibria are so extreme, the equilibrium will generally be in mixed strategies that are typically not directly observable. A small literature exists that addresses the use of mixed strategies by players in games. This literature considers two questions: Do players employ mixed strategies? If so, are they choosing the optimal mixed strategies?

The first question has been considered in several ways. Merolla, Munger, and Tofias (2005) note that when mixed strategies are used, a player would gain by altering the chosen action once the opponent's strategy is known. They argue that in the 2000 US presidential election, Gore used mixed strategies, since he would have gained from reallocating resources had he known what allocations Bush was making across states. To make this argument, they must determine the effect that reallocating expenditures would have on the outcome in different states. They base this analysis on comparing outcomes to pre-election polls. Chiappori, Levitt, and Groseclose (2002) argue that in a game in which players use mixed strategies and make a number of moves, each move is a random realization of the strategy. Thus, their choices should exhibit independence over time. They show that goalies and kickers in soccer matches behave consistently with using mixed strategies during penalty kicks since their choices show no serial correlation.

The second question, optimality of the mixed strategies, is considered by Chiappori, Levitt, and Groseclose (2002) in the soccer context and Walker and Wooders (2001) with respect to tennis serves. Both sets of authors compute what the optimal mixtures would be, and both conclude that players choose strategy mixes close to the optimal ones. Walker and Wooders find that more experienced players choose mixed strategies closer to the optimal ones but all players mix more than would be optimal.

The approaches used above cannot be applied in many settings including political campaigns. Often there is only one realization of a candidate's strategy, not a series of observations as with tennis serves or soccer kicks. Strategy choices cannot be easily related to outcomes. For example, polling data is not as extensive for lower-level political races as it is for presidential elections. The necessary condition we derive gives a novel way of testing whether the Blotto framework is appropriate in various contexts even if the candidates use mixed strategies by directly examining the actions of players to see if they ever choose symmetric actions across battlefields. With this approach, it is not necessary to make empirical inferences about the effects of strategy choices on outcomes.

For many of the applications, getting the appropriate data on players' strategic choices would be difficult. For the case of political campaigns, however, detailed data exists on political advertising, one important strategic variable available to candidates. Recent data on political campaigns in the United States from the Campaign Media Analysis Group (CMAG) contains extensive information on political advertisements at the individual ad level including content, length, time, station of airing, and estimated cost. The data, sold to candidates and news organizations desiring real-time information about campaign activity, are released for academic use after a lag of several years. The only data currently available for a non-Presidential year are from 2002. Despite some shortcomings, that candidates purchase these data indicates that they contain useful information about strategic choices.[11]

In general, the candidates seem to use strategies which are not consistent with what the Blotto model would predict, with advertising allocations much more equally divided across markets.[12] This supports the conclusion that equal allocations might be explained by a more continuous game of allocative mismatch, such as a probabilistic voting model as in Fletcher and Slutsky (2009) or a model using a contest success function.[13] Of course, this evidence does not argue that the Blotto model is not appropriate in any of the other settings to which the model might apply.

The formal model is described in Section II. Pure strategy equilibria are considered in Section III. Some properties of mixed strategy equilibria are derived in Section IV. The empirical analysis related to political campaigns is given in Section V and conclusions are discussed in Section VI. All proofs are contained in an Appendix.

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[11] The data are only for broadcast and cable television in the 100 largest media markets, and do not include advertising on other media such as print or radio. Some aspects of the data are questionable, particularly the price component. See Goldstein and Freedman (2002) for a detailed discussion of the CMAG data and its limitations.

[12] This situation differs from the outcome that is observed in Presidential elections, where candidates allocate asymmetrically across states. In the former, the magnitude of the win in a battlefield (market) matters because it is the sum of votes over all the markets in a state that matters. In the latter, the magnitude of a player's win in a battlefield (state) is unimportant, since a player wins all the Electoral College votes of the state if he wins that state by the narrowest of margins.

[13] Skaperdas (1996) axiomatizes several contest success functions that have been widely used in the literature.

(II) The Model

Consider a contest between two players A and B, each of whom divides resources between two battlefields denoted m and n. Each battlefield has three sub-battlefields denoted 1, 2 and 3. Sub-battlefield (i, j) denotes sub-battlefield i of battlefield j. Each sub-battlefield possesses two characteristics: importance and advantage. Importance may be the size of the sub-battlefield, but could more generally relate to its significance to the players. The importance of sub-battlefields are denoted by m_i and n_i, i = 1, 2, 3. The overall importance of a battlefield is assumed to be the sum of the importance of the sub-battlefields, with $M = \Sigma m_i$ and $N = \Sigma n_i$. The two players have the same evaluation of the importance of sub-battlefields. Advantage relates to the extent to which the characteristics of a sub-battlefield favor one of the players over the other. For convenience, advantages to A are treated as positive while advantages to B are treated as negative. Sub-battlefield 1 in each battlefield favors B with the size of the advantage denoted as $-\theta_m$ and $-\theta_n$, while sub-battlefields 3 favor A with the size of the advantage denoted as γ_m and γ_n. Sub-battlefields 2 are neutral, with neither player having an advantage before the battle begins.

The total resources available to the two players are R_A and R_B, which we assume are exogenously determined. The players can allocate their resources across the two battlefields with x_m and x_n denoting the allocations of player A and y_m and y_n denoting the allocations of player B. Thus, the two players face resource constraints

$$x_m + x_n = R_A \text{ and } y_m + y_n = R_B \tag{1}$$

An allocation to a battlefield goes to all the sub-battlefields in that battlefield in the same amount. Resources allocated to a battlefield by a player will shift the advantage toward that player. The effectiveness of resources in shifting the advantage are given by the functions h_{ij} in sub-battlefield i of battlefield j, where

$$h'_{ij} > 0, h''_{ij} \leq 0, h_{ij}(0) = 0 \qquad (2)$$

For example, in sub-battlefield (1, m), the post-battle advantage to player B is $-\theta_m + h_{1m}(x_m) - h_{1m}(y_m)$, while in sub-battlefield (3, n), the post-battle advantage to player A is $\gamma_n + h_{3n}(x_n) - h_{3n}(y_n)$. Figure 1 shows the post-battle situation in the two battlefields.

This specification generalizes the standard Colonel Blotto assumption that effectiveness is linear in expenditures. Here, we allow for the possibility of diminishing marginal effectiveness. In addition, specifying different functions for each sub-battlefield allows the effectiveness of allocations to differ across sub-battlefields. This could be due to differences between the sub-battlefields in aspects such as size and terrain. We continue to assume that the effectiveness functions are the same for both players and are independent of the level of the other player's allocation.

Any sub-battlefield is won by the player with the post-battle advantage. Let I denote the post-battle advantage to player A. Then the outcome in any sub-battlefield is specified as:

$$V(I) = \begin{cases} 1 \text{ if } I > 0 \\ 1\backslash 2 \text{ if } I = 0 \\ 0 \text{ if } I < 0 \end{cases} \qquad (3)$$

The overall payoff to Player A is

$\hat{V}(x_m, x_n, y_m, y_n) = m_1 V[-\theta_m + h_{1m}(x_m) - h_{1m}(y_m)] + m_2 V[h_{2m}(x_m) - h_{2m}(y_m)] + m_3$ $V[\gamma_m + h_{3m}(x_m) - h_{3m}(y_m)] + n_1 V[-\theta_n + h_{1n}(x_n) - h_{1n}(y_n)] + n_2 V[h_{2n}(x_n) - h_{2n}(y_n)] +$

$n_3 V[\gamma_n + h_{3n}(x_n) - h_{3n}(y_n)]$ (4)

The players simultaneously make their allocation decisions, with Player A choosing x_m and x_n to maximize $\hat{V}$ and player B choosing y_m and y_n to minimize $\hat{V}$, subject to the resource constraints (1).

Consider the following settings that fit within the context of this model.

Military campaigns

First, to extend the classic Colonel Blotto warfare model to incorporate imperfect targeting, consider a military campaign between army A and army B. Two battlefields, m and n, are contested. Army A allocates x_m soldiers to battlefield m and x_n soldiers to battlefield n, while army B allocates y_m and y_n soldiers across the battlefields. In each battlefield, the armies face off across a linear battlefront, with the terrain giving advantage to one army or the other at different points along the battlefront. Sub-battlefields 1 and 3 denote points along the front that are advantageous to army B and army A, respectively, while sub-battlefields 2 are neutral, favoring neither army. For example, one location in battlefield m might have boulders on army B's side of the front, giving that army's soldiers convenient cover. The magnitude by which army B gains from this cover is represented by $-\theta_m$. The importance parameter m_1 could be based on the size of this location or on its intrinsic value to the armies. Such points of advantage need not be geographically contiguous, so that it would be difficult for an army to send soldiers only to neutral points or to those points where that army has an advantage. Thus, each army is unable to perfectly target its troops to areas with a given advantage. The effectiveness of troops, represented by the h_{ij} functions, could differ across sub-battlefields because of differences in such factors as ease in communications or resupply.

Advertising campaigns

A second application of the model is in advertising campaigns, either between two firms competing for sales, or two political candidates competing for votes. In the product advertising case, the firms do not compete by changing the characteristics of the products, but through advertising. Consumers differ in which product they prefer, with some being indifferent. For example, a few years ago the University of Florida switched from selling only Coke products to selling only Pepsi products on campus. Some at the University were ecstatic about the change, others were outraged, and some, like the authors, were completely indifferent. The Coca-Cola Company and PepsiCo do not attempt to win consumers over by changing the formulas of Coke and Pepsi, but instead have advertising wars. As in Golman and Page (2009), the product advertising case is winner-takes-all if a store has shelf space for only one firm's product.[14] Another possibility is advertising to institutional users that will sign exclusive contracts to use one firm's good. Examples of this would be hospitals buying all their cleaning supplies from a single supplier, or schools buying all their milk from a single producer because of the influence of advertising.

In the political advertising example, the two candidates have platforms that are fixed prior to the beginning of the campaigns. The campaign of each candidate is designed to alter voter preferences in favor of that candidate. Since no voter is likely to

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[14] If some customers use market ranking as a signal of quality, then a discontinuous payoff would result similar to the winner-takes-all setting. A relatively small increase in advertising might increase the firm's market ranking, then induce a large increase in sales through the quality perception effect.

prefer one candidate's position over that of the other on every issue, ads might be designed to change voter preferences in other ways. For example, candidates might attempt to change the saliency of different issues in the minds of the voters, or focus on non-issue factors such as character or competence.[15]

The battlefields in either advertising case would be media markets m and n. In the political case, these are different media markets in the same electoral district, such as two cities in the same state for a Senate race. Submarkets 1 represent consumers with a preference for product B or partisans for candidate B, while submarkets 3 are the individuals preferring product or candidate A. Submarkets 2 are the neutral consumers or voters. Firms or candidates allocate advertising dollars x and y across markets, but cannot perfectly target who will see a given ad within a market. Since all sales and votes are equally important,[16] the importance parameters m_i and n_i simply represent numbers of consumers or voters. θ_j and γ_j are the preference intensities of the partisans for B and A, respectively. The h_{ij} are the advertising effectiveness functions in the various submarkets. Ads might have different effectiveness in different media markets because of different television viewing habits or differences in ad prices between large and small markets. Additionally, individuals with initially strong preferences toward a product or candidate might respond differently to ads than do initially neutral individuals.

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[15] In the political context, maximizing the probability of winning rather than expected plurality is considered preferable, but is often not used as it is significantly more complicated to analyze. A number of papers including Aranson, Hinich and Ordeshook (1974), Ledyard (1984), Snyder (1989), Duggan (2000) and Patty (2005) have considered the relation between outcomes under the two objectives in a spatial voting context. In some, perhaps special, circumstances, they are equivalent. It should be noted that expected plurality maximization is not only more convenient but in some circumstances can be justified as more realistic. Candidates with a low probability of winning may desire to lose by as small a margin as possible.

[16] In the political setting, this would be true in a statewide or district-wide race. Obviously, all votes are not created equal in an Electoral College setting.

Quality competition among providers of education

In the final example, two private school providers compete for students by varying the quality of education in two districts, m and n. Provider A is a religious organization and provider B is a secular firm. Quality differences are achieved by A via allocations x_m and x_n, and by B via y_m and y_n. The providers wish to maximize the total number of pupils they serve because of economies of scale arising from such factors as reduced textbook prices for large orders or fixed administrative costs.

Different types of parents have underlying preferences about sending their children to secular or religious schools, while some parents are indifferent. These underlying preferences θ_i and γ_j could be overcome with large enough quality differentials. Quality differentials are difficult to target toward pupils of one type and not another. 1, 3 and 2 are the parent types preferring secular or religious education and the neutrals, respectively. h_{ij}, the effectiveness of dollars spent on quality, might differ across communities because dollars buy different amounts of quality due to differences in land costs and prevailing wage rates, or due to differently sized physical plants. All of these factors would likely be correlated with community size. The effectiveness functions might also differ across parental types, as these may have different sensitivities to changes in quality. The importance parameters would generally relate to the numbers of the different types in each community but might also incorporate desires by the schools to have students with certain skills (academic or athletic).

Within the general structure specified above, we make some notational conventions and some technical assumptions to assure that the analysis is interesting.

Without loss of generality, we can rename the players so that A has at least as much resources as B (i.e. $R_A \geq R_B$). In addition, we can rename the battlefields so that n is at least as important as m ($N \geq M$).

If the player advantages in sub-battlefields 1 and 3 were so large as to be insurmountable by any allocation of resources by the opponent not favored in them, then the game would reduce to a single battlefield, and the extension would not be interesting. Thus, we assume that all sub-battlefields are in play. That is, each player has sufficient resources to potentially win those sub-battlefields in which the other player has an advantage:[17]

$$\theta_j < h_{1j}(R_A) \text{ and } \gamma_j < h_{3j}(R_B),\ j = m, n \qquad (5)$$

If the resource differences were very large, one player could overwhelm the other on all sub-battlefields. To rule this out we assume that the resource difference between the players is not very large relative to the sub-battlefield advantages:

$$\theta_j > h_{1j}(R_A - R_B) \text{ and } \gamma_j > h_{3j}(R_A - R_B),\ j = m, n \qquad (6)$$

In essence, allocating just the difference in resources available to the players to a battlefield is not enough to sway the outcome of any non-neutral sub-battlefield. Note that (5) and (6) with respect to γ_j imply that $R_A < 2R_B$.

Next, if there is an exact balancing of the importance of certain sub-battlefields, then knife-edge equilibria may result. For example, if $m_2 = n_2$, multiple equilibria may exist, because the gains and losses from small changes in allocations across the battlefields that only change the outcomes on the neutral sub-battlefields cancel out.

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[17] Note that other sub-battlefields could exist with very high advantages, for example, with $\gamma_{kj} > h_{kj}(R_B)$ but those sub-battlefields will always be won by the favored player. This, they need not be considered by players when setting strategies.

Such equilibria are non-generic, arising only on measure zero sets in the parameter space. The following condition rules out this type of equilibrium:

$$\sum_{i=1}^{3}(t_i n_i + s_i m_i) \neq 0 \text{ when } t_i \text{ and } s_i \text{ each take any values from the set} \qquad (7)$$

(0, 1, -1, ½, -½) but are not all zero.

The expression in (7) is a general formula for the change in $\hat{V}$ due to changes in a player's strategy. Either a strategy change has no effect in a sub-battlefield, or it causes victory in the sub-battlefield to switch from one player to the other, or it creates or breaks a tie in the sub-battlefield. One implication of this assumption is that $M \neq N$, so that the two battlefields cannot be exactly equal in importance. Therefore, given the naming convention above, $M < N$ must hold. Another implication is that $n_2 \neq m_2$, so the importance of the initially neutral sub-battlefields cannot be exactly equal.

(III) Pure Strategy Equilibria

In a Colonel Blotto game with perfect targeting, a pure strategy equilibrium generally exists when there are only two battlefields, with both contestants expending all their resources in the larger battlefield. With imperfect targeting, even though the players have only two alternatives on which to spend resources, there are really more than two battlefields and pure strategy equilibria typically do not exist. To analyze this, we begin with Lemma 1, which rules out interior pure strategy equilibria.

Lemma 1: Given parameter restrictions (5) – (7), there do not exist any pure strategy equilibria with $0 < x_m < R_A$ and $0 < y_m < R_B$.

To specify the conditions for pure strategy equilibria, consider the case when $R_A = R_B \equiv R$, and define the cutoff values $\bar{y}$ and $\bar{\bar{y}}$ by $h_{3m}(\bar{y}) = \gamma_m$ and $h_{1n}(R - \bar{\bar{y}}) = h_{1n}(R) - \theta_n$. When $x_m = 0$, A wins sub-battlefield (3, m) if $y_m < \bar{y}$ and B wins if $y_m > \bar{y}$, while A wins sub-battlefield (1, n) if $y_m > \bar{\bar{y}}$ and B wins if $y_m < \bar{\bar{y}}$. Similarly, let $\bar{x}$ and $\bar{\bar{x}}$ be defined by $h_{1m}(\bar{x}) = \theta_m$ and $h_{3n}(R - \bar{\bar{x}}) = h_{3n}(R) - \gamma_n$. When $y_m = 0$, A wins sub-battlefield (1, m) if $x_m > \bar{x}$ and B wins if $x_m < \bar{x}$, while A wins sub-battlefield (3, n) if $x_m < \bar{\bar{x}}$ and B wins if $x_m > \bar{\bar{x}}$. Given assumption (5), $\bar{y}$, $\bar{\bar{y}}$, $\bar{x}$, and $\bar{\bar{x}}$ all lie between 0 and R.

Theorem 1: Under assumptions (1) – (7) and the convention that the battlefields are named so that M < N, the only possible pure strategy equilibrium is $x_m = y_m = 0$. This equilibrium exists if and only if $R_A = R_B$, $m_2 < n_2$, and both of the following conditions hold:

(A) Either $\bar{y} < \bar{\bar{y}}$ and $\tfrac{1}{2}n_2 > \tfrac{1}{2}m_2 + m_3$ or $\bar{y} \geq \bar{\bar{y}}$ and $n_1 + \tfrac{1}{2}n_2 > \tfrac{1}{2}m_2 + m_3$

(B) Either $\bar{x} < \bar{\bar{x}}$ and $\tfrac{1}{2}n_2 > m_1 + \tfrac{1}{2}m_2$ or $\bar{x} \geq \bar{\bar{x}}$ and $\tfrac{1}{2}n_2 + n_3 > m_1 + \tfrac{1}{2}m_2$

From Theorem 1, when each player has sufficient resources which differ from those of the other player but not by too large an amount, then no pure strategy equilibrium exists. It never exists when $R_A \neq R_B$. When $R_A = R_B$, it exists only under further restrictions on the parameters of the model. To gain some intuition behind this result, recognize that given the budget constraint, each player has a one-dimensional strategy space: how much is allocated to battlefield m. Fixing the opponent's allocation, a player's payoff is nonmonotonic over this strategy space. As a player allocates more resources toward m, sub-battlefields in m will switch allegiance to that player, while those in n switch away at different allocation levels. Because of this non-monotonicity, at least one player will have an improving deviation over any pair of pure strategy allocations, one for each player.

The following result helps in understanding the conditions determining the relations between $\bar{y}$ and $\bar{\bar{y}}$ and $\bar{x}$ and $\bar{\bar{x}}$, at least in one special case. Assume that effectiveness is the same in the sub-battlefields of each battlefield, with $h_{im}(z) = h_m(z)$ and $h_{in}(z) = h_n(z)$, all i and z. Further, assume that battlefield m is less important than n because it is smaller, and hence that allocations are more effective there with $h_m(z) \geq h_n(z)$, all z.

Lemma 2: If $\gamma_m < \theta_n$, or if $\gamma_m = \theta_n$ and $h''_j < 0$, then $\bar{y} < \bar{\bar{y}}$. If $\theta_m < \gamma_n$, or if $\theta_m = \gamma_n$ and $h''_j < 0$, then $\bar{x} < \bar{\bar{x}}$.

Therefore, if the more important battlefield n also has larger advantages ($\theta_n > \gamma_m$ and $\gamma_n > \theta_m$), then from Lemma 2 and conditions (A) and (B) of Theorem 1, $\frac{1}{2}n_2 > \frac{1}{2}m_2 + m_3$ and $\frac{1}{2}n_2 > m_1 + \frac{1}{2}m_2$ must hold at a pure strategy equilibrium. Together these imply that $n_2 > M$. Not only is battlefield n more important overall, its neutral sub-battlefield must be more important than all of battlefield m. In this case, both players put all their resources in the more important battlefield and devote no resources to the less important one. If the advantage relations do not hold, the conditions on n_2 relative to the importance of sub-battlefields in m are still sufficient but are not necessary for the pure strategy equilibrium to exist. It is worth noting that in some settings, it is plausible to have larger advantages in the more important battlefield. For example, consider the advertising example, where the more important battlefield is simply the larger market. The advantages can be interpreted as how much resources must be expended to shift an individual's decision. In larger communities, the price of an advertisement may be larger, so that more must be spent to have an individual see the same number of ads.

Pure strategy equilibria may exist if one player's resources are significantly larger than the other with assumption (6) violated. For example, with enough resources, player A could set $x_m = x_n = \frac{1}{2}R_A$ and, no matter what B did, win all sub-battlefields in battlefield m provided $-\theta_m + h_{1m}(\frac{1}{2}R_A) - h_{1m}(R_B) > 0$, and all sub-battlefields in battlefield n if $-\theta_n + h_{1n}(\frac{1}{2}R_A) - h_{1n}(R_B) > 0$.

Although the formal results in Theorem 1 apply to a situation with only two battlefields, the intuition that pure strategy equilibria are unlikely to exist carries over to more than two battlefields. For a pure strategy equilibrium to exist in a setting of K battlefields, the equilibrium strategies restricted to any pair must be an equilibrium in that pair holding fixed allocations to all the other battlefields. Let R_A^{ij} and R_B^{ij} be the sum of resources allocated to battlefields i and j by the two players. If the players have total resources which do not differ much, then, on at least one pair of battlefields i and j, the resources R_A^{ij} and R_B^{ij} will satisfy assumptions (5) and (6) with an equilibrium then unlikely on the pair.

(IV) Mixed Strategy Equilibria

In most circumstances, no pure strategy equilibrium exists and the players use mixed strategies. Finding the equilibrium mixed strategies in general is difficult. In Theorem 2 below, we are able to find a necessary condition on the equilibrium mixed strategies under the assumption that players have the same resources. To specify this, let $E(x_m)$ and $F(y_m)$ denote the cumulative distribution functions for the mixed strategies of A and B, respectively. As cdfs, $E(x_m)$ and $F(y_m)$ are nondecreasing and right-hand continuous and can be divided into continuous parts $E^*(x_m)$ and $F^*(y_m)$ and countable sets of values at which positive probability is placed. Let $e^a(x_m)$ and $f^a(y_m)$ denote the probability placed on the specific values x_m and y_m if they are mass points. From right-hand continuity, $E(x_m) = \lim_{\delta \to 0} E(x_m - \delta) + e^a(x_m)$ and $F(y_m)$
$= \lim_{\delta \to 0} F(y_m - \delta) + f^a(y_m)$.

Consider the case in which the players have the same resources $R \equiv R_A = R_B$. Consider sub-battlefield (3, m), which favors player A. A wins the sub-battlefield if $\gamma_m + h_{3m}(x_m) - h_{3m}(y_m) > 0$. There exists an $\bar{x}$ such that $\gamma_m + h_{3m}(x_m) - h_{3m}(R) > 0$ iff $x_m > \bar{x}$. Since $\bar{x} < R$ must hold, there exists an $\alpha(\gamma_m) = R - \bar{x}$ with $0 < \alpha(\gamma_m)$. Directly, $\alpha(\gamma_m)$ is defined by $h_{3m}(R - \alpha(\gamma_m)) \equiv h_{3m}(R) - \gamma_m$. $\alpha(\gamma_m)$ defines a lower bound on x_m such that A wins sub-battlefield (3, m) no matter how much B spends in battlefield m. That is, for any y_m and all $x_m > R - \alpha(\gamma_m)$, $\gamma_m + h_{3m}(x_m) - h_{3m}(y_m) > 0$. Similarly, $\alpha(\gamma_n)$, defined by $h_{3n}(R - \alpha(\gamma_n)) \equiv h_{3n}(R) - \gamma_n$, gives a lower bound on x_n such that for all $x_n > \alpha(\gamma_n)$ and any y_n, $\gamma_n + h_{3n}(x_n) - h_{3n}(y_n) > 0$. For sub-battlefields 1 of both battlefields, which lean to B, we can define $\alpha(\theta_m)$ by $h_{1m}(\alpha(\theta_m)) \equiv \theta_m$ and $\alpha(\theta_n)$ by $h_{1n}(\alpha(\theta_n)) \equiv \theta_n$. In this case, for all $x_m < \alpha(\theta_m)$ and any y_m, $-\theta_m + h_{1m}((x_m) - h_{1m}(y_m) < 0$, so B definitely wins (1, m). Similarly, for all $x_n < \alpha(\theta_n)$ and any y_n, $-\theta_n + h_{1n}(x_n) - h_{1n}(y_n) < 0$, so B definitely wins (1, n). Since $h_{1i}(R) > \theta_i$ by assumption (5), $\alpha(\theta_i) < R$ must hold. Let $\alpha \equiv \min[\alpha(\theta_m), \alpha(\theta_n), \alpha(\gamma_m), \alpha(\gamma_n)]$. From the construction of $\alpha(\theta_i)$ and $\alpha(\gamma_i)$, it immediately follows that $\alpha < R$. In addition, we assume

$$R < 2\alpha \tag{8}$$

Clearly, it follows from (8) that $R - \alpha < \frac{1}{2} R < \alpha$. If either player chooses an allocation in the interval $(R - \alpha, \alpha)$, then, no matter what its opponent does, that player cannot win either of the sub-battlefields favorable to the opponent, but also cannot lose either sub-battlefield in which that player is favored. Only the neutral sub-battlefields are in play. On the other hand, if an allocation is chosen outside the interval, there are potential gains and losses in the other sub-battlefields. For example, if x_m is below $R - \alpha$, this means that x_n is above α. In this case, if y_m is large, A could lose sub-battlefield (3, m), while B could lose sub-battlefield (1, n).

Given a value of R, this assumption in effect puts lower bounds on the magnitudes of the θ_i and γ_i. These advantages cannot be too large relative to R from assumption (5). From (8), they also cannot be too close to 0. Note that (8) in at least some circumstances strengthens (6). If the h_{ij} are linear, then $\alpha(\theta_i) = c_{ij}\theta_i$ and $\alpha(\gamma_i) = c_{ij}\gamma_i$, for some positive constants c_{ij} so $\alpha \equiv \min(c_{1m}\theta_m, c_{1n}\theta_n, c_{3m}\gamma_m, c_{3n}\gamma_n)$. Hence, for $R_A = R_B$, (6) only imposes $\alpha > 0$ while (8) imposes the more restrictive $\alpha > \frac{1}{2} R$.

Theorem 2: Assume $R_A = R_B$ and parameter restrictions (5) – (8). Either $E(\alpha) - e^a(\alpha) = E(R - \alpha)$ or $F(\alpha) - f^a(\alpha) = F(R - \alpha)$, so that at least one player never expends resources in the open interval $(R - \alpha, \alpha)$.

Although multiple mixed strategy equilibria may exist and it is difficult analytically to find the exact equilibria, Theorem 2 gives an important characterization of any mixed strategy equilibrium, at least in circumstances when the resources available to the two players are the same. In no equilibria do both players have any probability of dividing their resources nearly equally between the two battlefields. This does not imply that even one of the players will, with positive probability, have an approximately equal division of expenditure between battlefields. In fact, if the game is symmetric, then neither player will ever have an approximately equal division.

Corollary 1: If $R_A = R_B = R$, $\theta_i = \gamma_i$, $n_1 = n_3$ and $m_1 = m_3$, then in any mixed strategy equilibrium $E(\alpha) - e^a(\alpha) = E(R - \alpha)$ and $F(\alpha) - f^a(\alpha) = F(R - \alpha)$.

The significance of these results depends upon the size of the interval $(R - \alpha, \alpha)$. For α near its lower bound of ½ R, the interval around ½ R is small. Only allocations with almost exactly equal spending in the two battlefields are ruled out. For α near its upper bound R, this interval is almost the entire set of allocations. Anything except for almost complete asymmetry in expenditures is ruled out. The size of this interval thus depends upon the magnitude of the initial advantages relative to the resources available to players. The more resources players have relative to these advantages, the more asymmetric must be the allocations of a player.

Throughout the analysis, we have assumed that the resources are exogenous and uncorrelated to the advantages θ_i and γ_i. However, there may be reasons for them to be positively or negatively correlated in some of the examples of the model. Consider the case of political campaigns. When the advantages are greater voters are less persuadable, so money will be less effective and donors may prefer to donate to other candidates in other races, yielding a negative correlation. On the other hand, in larger battlefields, more money is raised but ads may be more expensive, which is similar in effect to having larger advantages, creating a positive correlation.[18] In general, assume that $\alpha = a + bR$ where the sign of b determines whether α and R are positively or negatively correlated. Then the interval becomes $((1 - b)R - a, bR + a)$ with the size of the interval $2a + (2b - 1)R$. The size of this interval declines in R for $b < ½$ and increases for $b > ½$. Only if b is near ½ and a is near 0 will the interval be very small for all values of R. Otherwise, for at least some R, it should be non-negligible in size.

Both Theorem 2 and Corollary 1 are knife-edge results, holding for situations with exactly equal resources for the players or exactly symmetric games. However, the essence of the results are likely to carry over to nearby situations with the resources of the players similar but not exactly equal, or to almost symmetric games. If the equilibrium correspondence of the game is upper hemi continuous, since every equilibrium at $R_A = R_B$ has one player placing no weight near equal division, then for a neighborhood of resource pairs around $R_A = R_B$, at least one player would put little probability weight in the interval $(R_A - \alpha, \alpha)$ or $(R_B - \alpha, \alpha)$. Similarly, for almost symmetric games, both players would put little weight in those intervals.

(V) Empirical Evidence in Political Campaigns

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[18] See Stratmann (2005) for a survey of the recent campaign contributions and campaign spending literatures, where he discusses both of these effects. Stratmann notes that when voter preferences are strong toward one candidate or the other, contributors are less likely to donate since campaign spending would have little effect in those markets. He also discusses the fact that spending is likely to buy very different amounts of advertising in different markets.

If the candidates' behaviors are consistent with the Blotto approach, then at least one candidate will use very asymmetric strategies across markets, regardless of whether the equilibrium is in pure or mixed strategies. To test this, we examine the data to see if candidates run substantially different numbers of advertisements across media markets. Note that strategies are not compared across candidates, but across markets for each candidate. Our advertising data include information only for the 100 largest broadcast markets in the United States. We initially include in our analysis any part of a market in a state with a political contest, even if only a tiny fraction of the market is in the state in question. The implications of this will be discussed below.

We first consider states in which there are only two markets or in which one market is much larger than all the rest of the markets combined. These are the situations most consistent with the formal model. The results above were derived for allocations between two markets. When more markets exist but one is very large relative to the others, it can be shown that a pure strategy equilibrium will still exist with the candidates only running ads in the large market.

Table 1 lists each of the eleven states consistent with these criteria. Many media markets cross state borders, so that the market may be divided between two, three, or even more states. The table includes all media markets, or parts of media markets, in the state, and lists the population of the market that is within that state in question. For example, the Washington, D.C. media market falls not only in the District of Columbia proper, but also in the states of Maryland, Virginia and West Virginia. The population listed in the table for Washington, D.C. includes only those individuals in the D.C. market who live in the state of Maryland. The fourth column in the table, "fraction of state," shows the fraction of the state's broadcast market population[19] that falls in the market in question. For each state, these fractions should sum to 1, with some allowance for rounding. The final column in the table, "fraction of market," shows the fraction of the market that falls in the state listed. For example, all of the Baltimore media market is within the state of Maryland, but only 45% of the D.C. broadcast market is in Maryland.

That states may have very small parts of large media markets along with other markets has interesting implications for our analysis. If ad prices are proportional to market population, and if two media markets have the same fraction of their populations in a particular state, then the cost to a candidate of reaching a single voter is the same in both markets even if the markets differ in size. However, if a state has a large fraction of one market and a small fraction of a second market, then the relative costs of reaching a single voter is proportional to these fractions.

First, consider the four states in our sample with only two markets or market portions: Maryland, New Jersey, New Mexico and Oregon. The markets in Maryland are of approximately equal size. Recall, however, that the Baltimore market is entirely in Maryland whereas less than half of the Washington, D.C. market is in Maryland. Thus, it is more than twice as expensive to reach a Maryland voter by buying ads in the D.C. market as in the Baltimore market.

The other three states have much larger differences in market size. In New Jersey, the New York market is about three times as large as the Philadelphia market but the fractions of each market in that state are about the same, so the cost of reaching a single New Jersey voter from each market are very similar. In New Mexico, the

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[19] We define the state's broadcast market population as the population living within one of the 100 largest broadcast markets. State residents living outside one of these broadcast markets are not included in our totals.

Albuquerque market is about nine times the size of the El Paso market. Only a fifth of the El Paso market, but nearly all of the Albuquerque market, is in New Mexico. This implies that the cost of reaching a voter by an ad there is perhaps four and a half times the cost in Albuquerque.

Finally, in Oregon, the Portland market has 228 times the population of the Spokane market. This very large difference is due to the fact that such a small fraction (less than 1%) of the Spokane market falls in the state of Oregon, whereas the entire Portland market is in Oregon. Thus, the cost of reaching an Oregon voter from Spokane is 100 times that of reaching a voter from Portland. Given this price differential, it would be consistent with almost any optimizing model --- not just Blotto --- for Oregon candidates to place no ads in Spokane, and this is in fact what we see in the data. For this reason, we think that it is appropriate to drop Spokane as a market in this state and hence to treat Oregon as having only one media market.

Second, consider the seven states with multiple markets but where one market is much larger than all the other markets in the state combined: Arizona, Colorado, Georgia, Illinois, Nebraska, New Hampshire, and Vermont. Several of the markets in these states will have very high relative advertising prices --- more than 20 times that in another market --- and will thus be dropped from the analysis. These include the portion of the Albuquerque market in Arizona (where the relative price would be 25 times that in Phoenix or Tucson) and Colorado (where the relative price would be at least 24 times that in Colorado Springs or Denver). By the same logic, Greenville and Jacksonville are dropped as Georgia markets, Evansville is dropped as an Illinois market, and Nebraska and Vermont reduce to having only a single market. In each of the dropped markets, no ads were run by any candidate.

Consider the remaining markets in the states of Georgia, Illinois, and New Hampshire. In each of these states, one market contains at least three quarters of the state's total broadcast population: the Atlanta market has 82 percent of Georgia's broadcast market population, while Chicago has 78 percent of the Illinois broadcast market population and Boston has 82 percent of New Hampshire's. If more than a quarter of the populations in these large markets are neutral before the campaign begins, the size of the neutral submarket will be larger than all the other markets in that state combined. These cases would fit within our theoretical result mentioned above, that a pure strategy equilibrium would exist where both candidates should allocate all their resources to the larger market.

Table 2 shows the total number of advertising occurrences by each candidate in the five states that have two effective markets. In Maryland and New Jersey, where the populations of the state's two markets are relatively close, the strategies are surprisingly symmetric, with a 60-40 or closer division of ads across the two markets. One candidate in Maryland had a 51 to 49 division and one in New Jersey divided the ads 52 to 48. Depending on how we define "even allocation," this fails to satisfy the result that the candidates cannot allocate evenly across the markets.

In the New Mexico Senate race, the ad buys are much more consistent with the pure strategy equilibrium of allocating all ads to the much larger market. One candidate ran no ads in the smaller El Paso market, and the other ran only 8 percent of his ads there. In the gubernatorial contest, both candidates ran non-trivial numbers of ads in El Paso (27 percent of total ads for one candidate and 15 percent for the other). This seems less consistent with Blotto and more consistent with a price effect, since the price of reaching one voter with an ad is four and a half times as high in El Paso as in Albuquerque.

The cases of Arizona and Colorado are especially interesting. In both, the effective markets fall almost entirely within the state, so that the costs of reaching a single voter are effectively the same. In the gubernatorial race in Arizona and the Senate race in Colorado, the two candidates ran almost the same total number of ads, indicating that they had similar sized budgets; this is consistent with the assumption in Theorem 2. The fraction of ads run in the smaller market varies across candidates from 35 to 44 percent of the total number. This could only be consistent with the Blotto results if fewer than a quarter of the voters in the large market were swing voters, so a pure strategy equilibrium does not exist, and if these divisions between markets were outside the zero probability intervals.

Table 3 shows the total number of advertising occurrences by each candidate in the three states with more than two effective markets, but where one market is much larger than the others combined. The only case that is consistent with Blotto is the gubernatorial race in New Hampshire. In that race, both candidates only ran ads in Boston. This cannot be viewed as a price effect, since the fraction of the Boston market in New Hampshire is almost equal to that of the Burlington market and is only twice that of the Portland market. On the other hand, in the Senate race in that state, the candidates ran ads in all three markets with the losing candidate, who ran almost 50 percent more ads in total than the winner, placing a significant number of ads in Burlington.

In Georgia and Illinois, allocations across markets are much more symmetric. Only one candidate --- Perdue, the winner of the Georgia race for governor --- ran more than half of his ads in the larger market. Even in this case, though, the allocation is much more equal across markets than the extreme asymmetry predicted by the Blotto model. In the Illinois gubernatorial race, Ryan ran almost the same number of ads in Champaign as in Chicago, while the number of ads that Blagojevich ran in Champaign was more than 70% of the number he ran in Chicago. They ran fewer but still significant numbers in the smaller but more expensive markets of Davenport, Paducah, and St. Louis. Again, these allocations are far more symmetric than those predicted by the Blotto model.

Next consider all the pairs of markets some candidate faces, even where states have more than two markets. Although the model does not directly address games with more than two markets, plausibly mixed strategies will still be used in such cases, based on results for standard Blotto games (Roberson (2006)). Our dataset contains 274 candidate-market pairs (so each observation is a market pair faced by a given candidate). 202 pairs involve candidates with relatively equal resources, where the total number of ads shown by one candidate is no more than twice that of the other. In 98 of these, at least one candidate allocates his advertising roughly equally (within 5% of an equal allocation or closer over the pair of markets). Figure 2 shows a histogram of the allocation of advertising occurrences in these 98 pairs.

The theory implies there should be no weight around 0.5, but the histogram clearly shows most of the density being about the point where the candidate's resources are split equally over the pair of markets.[20, 21] Because the width of the zero probability

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[20] If both candidates allocate approximately equal resources across the pair, both candidates will appear twice in the histogram, as each candidate's opponent is in the interval near 0.5. This seems justifiable since

interval depends in part upon unobservable factors, it could be that the interval is small and, therefore, the observations in Figure 2 that are near 0.5 are actually outside the interval. However, the size of this interval will generally vary with the resources available to the candidates and will not be very small for all levels of resources. To consider this, we order the 98 market pairs in Figure 2 by the candidate's level of resources utilized for the pair and then draw separate histograms for the top 49 and the bottom 49 as shown in Figure 3. In both cases, the opponents of candidates who divide their advertising almost equally between markets also divide equally with high frequency. For most parameter values, for at least one of the top or bottom halves, the interval will not be small, strengthening the plausibility that the data contradict the result in the theory.

Candidates could also vary their strategies over time, so that they show more ads in one market than another at one point in the campaign, and change that allocation at a later point. The result might make the aggregate allocations for the campaign appear to be more equal than at any point in time. Figure 4 shows the allocation across market pairs during the first week of October and again during the first week of November. Again, note that the bulk of the distribution is around the point where half the resources are devoted to each market.

Overall, the evidence seems inconsistent with the theory. In each case, the differences in the strategies undertaken in different markets by the candidates are

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either allocating around 0.5, given the other's allocation, violates the theory. Alternatively, histograms including candidates only once show less weight about 0.5, but still more weight about 0.5 than at less symmetric allocations.

[21] These results hold whether we consider the number of ads (as shown) or total advertising time, or whether we consider all ads on behalf of the candidate (as shown) or only ads sponsored by the candidates. Histograms representing allocations at different points in the campaigns also have the same shape as those shown, suggesting that candidates are not mixing over time.

considerably smaller than the differences that should arise at least some of the time from using mixed strategies

(V) Conclusions

Standard Colonel Blotto games assume perfect targeting of battlefields. Our extension of the model adds the realistic feature that players can target battlefields only imperfectly, so that actions targeted toward one battlefield can impact other battlefields. In particular, we study a polar case where some battlefields are grouped together, and cannot be targeted separately. Each of them receives the same allocation, although they may be impacted differently by it. The battlefields in a grouping may differ in their importance and in whether one of the players has an advantage over the other in them. We show that pure strategy equilibria will exist only in extreme circumstances and when they do, the players will choose very asymmetric allocations across groupings.

We then derive a necessary condition for mixed strategy equilibria when the players have the same amount of resources to allocate, and this amount lies between upper and lower bounds that depend upon the magnitude of the advantages that one player has over the other. This condition shows that at least one player must utilize asymmetric strategies, putting no probability weight in an interval around equal allocations to the two battlefields. When we consider how political candidates allocate their advertising expenditures, we do not observe this asymmetric behavior.

The use of asymmetric strategies by players arises largely from the discontinuous nature of payoffs in Colonel Blotto type games: a small change in one player's allocation to a sub-battlefield can induce a large change in payoffs. For players, such as political candidates, to act in a much more symmetric manner indicates that their behavior is smoothed out in some way. One possibility is that there are more than three sub-battlefields in each battlefield. Generalizing this to having a finite number of sub-battlefields, each with strictly positive importance, would not eliminate the discontinuities. However, if there were a continuum of sub-battlefields, each of which would then have infinitesimal importance, the discontinuity might disappear and the players would behave more symmetrically. A second possibility is that there is some randomness in the outcomes, so that a player wins a sub-battlefield with some probability even though putting fewer resources there. This can also smooth out the behavior and allow for relatively symmetric pure strategy equilibria to exist. As shown in Fletcher and Slutsky (2009), such a model is consistent with the data on campaign advertising.[22]

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[22] In the voting context, adding noise to a Blotto model leads to probabilistic instead of deterministic voting. This has been studied in a variety of contexts. See, for example, Coughlin (1992) and Lindbeck and Weibull (1987).

Appendix

Proof of Lemma 1: First, we consider a situation with $0 < x_m = y_m < R_B$, and show this cannot be an equilibrium, with a similar argument for $x_n = y_n$. Since $h_{2m}(x_m) = h_{2m}(y_m)$, each player gets ½$m_2$ from sub-battlefield (2, m). If the players are not tied in any other sub-battlefield, then a slight increase in x_m or y_m would gain the player changing strategy the other ½m_2 and not affect the outcome in any other sub-battlefield. Thus, this situation would not be an equilibrium. Consider if there is also a tie in a sub-battlefield of battlefield n. If $R_A = R_B$, then $x_n = y_n$ would also hold and each player would also get ½n_2 as well as ½m_2. Since $m_2 \neq n_2$ from assumption (7), an increase in x_m or y_m would benefit the player changing strategy if $m_2 > n_2$, while lowering x_m or y_m would be beneficial if $m_2 < n_2$. If $R_A > R_B$ and $-\theta_n + h_{1n}(R_A - x_m) - h_{1n}(R_B - y_m) = 0$ then each player would get ½n_1 from sub-battlefield (1, n). Again from assumption (7), $n_1 \neq m_2$, so a change in x_m or y_m would exist that increases the expected payoff of the player changing strategy. Thus, an equilibrium cannot exist on the interior with $x_m = y_m$ or with $x_n = y_n$.

Second, consider a situation with $x_m > y_m$ and $x_n < y_n$, with a symmetric argument when $x_m < y_m$ and $x_n > y_n$. Consider the change ΔV^B in player B's expected payoff from switching from y_n to $\overline{y_n} = x_n$. Then $\overline{y_m} = R_B - x_n \le R_A - x_n = x_m$. Hence, for the sub-battlefield (3, m), $\gamma_m + h_{3m}(x_m) - h_{3m}(y_m) > 0$ and $\gamma_m + h_{3m}(x_m) - h_{3m}(R_B - x_n) > 0$, so A continues to win that sub-battlefield. Similarly, sub-battlefield (1, n) is won by B before and after the change. The winner of sub-battlefield (2, m) does not change if $R_A > R_B$, but goes from A winning to a tie if $R_A = R_B$. The post-battle advantage to B in sub-battlefield (1, m) increases because of the change, so B does at least as well. The post-battle advantage to A in sub-battlefield (3, n) increases because of the change, so B's payoff in that sub-battlefield cannot increase. Sub-battlefield (2, n) goes from B winning to a tie. Thus: $\Delta V^B = -\frac{1}{2}n_2 + t_3n_3 + s_1m_1 + s_2m_2$, where $t_3 \in (0, -\frac{1}{2}, -1)$, $s_1 \in (0, \frac{1}{2}, 1)$ and $s_2 \in (0, \frac{1}{2})$.

Now consider B staying at y_n and A changing x_n to $\bar{x}_n = y_n$. The effects on battlefield n are exactly the same as those above, since $h_{in}(x_n) - h_{in}(\bar{y}_n) = h_{in}(\bar{x}_n) - h_{in}(y_n) = 0$. As above, there is no change in the outcome in sub-battlefield (3, m), and sub-battlefield (2, m) is affected iff $R_A = R_B$. Therefore: $\Delta V^A = \frac{1}{2}n_2 - t_3n_3 - s_2m_2 - \hat{s}_1m_1$, where $\hat{s}_1 \in (0, \frac{1}{2}, 1)$.

To complete this step we compare s_1 to $\hat{s}_1$. Before any change, $-\theta_m + h_{1m}(R_A - x_n) - h_{1m}(R_B - y_n)$ was ambiguous since $h_{1m}(R_A - x_n) - h_{1m}(R_B - y_n) > 0$. When y_n changes, this term equals $-\theta_m + h_{1m}(R_A - x_n) - h_{1m}(R_B - x_n)$ while when x_n changes, this term equals $-\theta_m + h_{1m}(R_A - y_n) - h_{1m}(R_B - y_n)$. Then $0 \leq h_{1m}(R_A - x_n) - h_{1m}(R_B - x_n) \leq h_{1m}(R_A - y_n) - h_{1m}(R_B - y_n) < h_{1m}(R_A - x_n) - h_{1m}(R_B - y_n)$ where the left inequality is strict iff $R_A > R_B$. The middle inequality follows from the assumption that $h''_{lm} \leq 0$ and is strict iff $R_A > R_B$ and $h''_{lm} < 0$. Thus, the effect in this sub-battlefield from lowering y_n is at least as big as that from raising x_n, making $\hat{s}_1 \leq s_1$. This means that $-\Delta V^A \leq \Delta V^B$. From assumption (7), $\Delta V^B \neq 0$. If $\Delta V^B > 0$, B could gain by lowering y_n. If $\Delta V^B < 0$ then $\Delta V^A > 0$ and A would gain by raising x_n. Thus, this situation cannot be an equilibrium.

Third, we consider the case where $x_m > y_m$ and $x_n > y_n$. Since $R_A < 2R_B$ by assumptions (5) and (6), at least one of $x_m < R_B$ and $x_n < R_B$ must hold. Without loss of generality we assume that $x_m < R_B$. Since $h_{im}(x_m) - h_{im}(y_m) > 0$ and $h_{in}(x_n) - h_{in}(y_n) > 0$, A wins sub-battlefields 2 and 3 in both battlefields. The outcome in sub-battlefield 1 in either battlefield cannot be a tie in equilibrium. If it were, using assumption (7), a small change in either x_m or y_m would raise the expected payoff of A or B, respectively. Nor can A win either sub-battlefield 1. If $-\theta_n + h_{1n}(x_n) - h_{1n}(y_n) > 0$, then an increase of y_m to $\overline{y}_m = x_m$ would gain B at least ½m_2 and risk losing nothing. Hence, this could not be an equilibrium. A necessary condition for this to be an equilibrium, then, is $-\theta_n + h_{1n}(x_n) - h_{1n}(y_n) < 0$. If $-\theta_m + h_{1m}(x_m) - h_{1m}(y_m) > 0$, then B wins no sub-battlefields in battlefield m. Reducing y_m to 0 cannot harm B in battlefield m. In battlefield n, $-\theta_n + h_{1n}(x_n) - h_{1n}(y_n) < 0$ must now hold. This follows since $h_{1n}(x_n) - h_{1n}(R_B) < h_{1n}(R_A) - h_{1n}(R_B) \leq h_{1n}(R_A - R_B) < \theta_n$ where the middle inequality follows from $h''_{1n} \leq 0$ and the right-hand inequality follows from assumption (6). Therefore, B clearly gains from this change. Hence, if this situation is to be an equilibrium, $V^A = m_2 + m_3 + n_2 + n_3$ and $V^B = m_1 + n_1$ must hold. Consider an increase in y_m to $\overline{y}_m = x_m$. Then B gains ½m_2. The outcomes in the other sub-battlefields of battlefield m do not change. In battlefield n, y_n is reduced so player A is more likely to win any sub-battlefield there. This change could only decrease B's payoff in sub-battlefield (1, m). However, $h_{in}(x_n) - h_{in}(R_B - x_m) \leq h_{in}(R_A - R_B) - h_{in}(0)$ since $h''_{in} \leq 0$ and, by assumption (6), $-\theta_n + h_{1n}(R_A - R_B) < 0$. Hence, $-\theta_n + h_{1n}(x_n) - h_{1n}(R_B - x_m) < 0$ and B continues to win that sub-battlefield. Overall, this change raises B's payoff, so this situation cannot be an equilibrium.

Thus, no situation with both players at an interior can be an equilibrium.

Q.E.D.

Proof of Theorem 1: From Lemma 1, a pure strategy equilibrium cannot have both players using interior strategies. Then, consider situations with at least one player at a boundary. First, we consider $R_A > R_B$ and show that no pure strategy equilibria exist with one of the players using a strategy on the boundary of the strategy space. The general approach used in the proof is to consider a boundary value for one player, find the best reply by the other, and then show that the original strategy is not a best reply to its best reply.

(i) Consider $x_m = 0$. For every value of y_m, B wins sub-battlefield (1, m) while A wins sub-battlefields (2, n) and (3, n). For any $y_m > 0$, B wins sub-battlefield (2, m). $y_m = 0$ is ruled out since with that allocation B gets $\frac{1}{2}m_2$ instead of m_2 and has the same outcome in all other sub-battlefields as $y_m = \varepsilon$. Thus, only the outcomes in sub-battlefields (3, m) and (1, n) change as y_m varies. With $\bar{y}$ defined by $h_{3m}(\bar{y}) = \gamma_m$, let $\bar{\bar{y}}_m$ be defined by $h_{1n}(R_B - \bar{\bar{y}}_m) = -\theta_n + h_{1n}(R_A)$. Both $\bar{y}$ and $\bar{\bar{y}}_m$ are positive and less than R_B from assumptions (2), (5), and (6). For $y_m < \bar{y}$, A wins sub-battlefield (3, m) and for $y_m > \bar{y}$, B wins that sub-battlefield. B wins sub-battlefield (1, m) when $y_m < \bar{\bar{y}}_m$, while A wins it when $y_m > \bar{\bar{y}}$. If $\bar{y} < \bar{\bar{y}}_m$, then any y_m such that $\bar{y} < y_m < \bar{\bar{y}}_m$ is a best reply by B with $V^A = n_2 + n_3$ and $V^B = M + n_1$. Given such y_m, consider $x_m = y_m + \varepsilon$. Then $V^A = m_2 + m_3 + n_2 + n_3$, which yields A a better payoff. If $\bar{\bar{y}}_m < \bar{y}$, B's best reply is either (a) $0 < y_m < \bar{\bar{y}}_m$ or (b) $\bar{y} < y_m \leq R_B$. It cannot be a best reply for $\bar{\bar{y}}_m \leq y_m \leq \bar{y}$, since B gets neither m_3 nor n_1. A does better with $x_m = y_m + \varepsilon$ getting $V^A = m_2 + m_3 + n_2 + n_3$. This exceeds A's payoff of $m_3 + n_2 + n_3$ from $x_m = 0$ when B's best reply is in (a). When B's best reply to $x_m = 0$ is (b), then $m_3 > n_1$ must hold. Since $\Delta V^A = m_2 + m_3 - n_1$ from playing $y_m + \varepsilon$ instead of 0, this is clearly positive, so A would not play $x_m = 0$ in response to B's best reply. If $\bar{y} = \bar{\bar{y}}_m$, a similar argument follows as when $\bar{y} > \bar{\bar{y}}_m$.

(ii) The argument for $x_m = R_A$ is identical to that for $x_m = 0$, reversing battlefields m and n.

(iii) Consider $y_m = 0$. For every x_m, A wins sub-battlefields (2, m) and (3, m). To evaluate sub-battlefield (1, n), note that $h_{1n}(R_A - R_B) \geq h_{1n}(R_A) - h_{1n}(R_B) > h_{1n}(R_A - x_m) - h_{1n}(R_B)$ where the first inequality follows from concavity of h_{1n}. Since $-\theta_n + h_{1n}(R_A - R_B) < 0$ from assumption (6), then concavity implies that $-\theta_n + h_{1n}(R_A - x_m) - h_{1n}(R_B) < 0$ and B always wins the sub-battlefield. B wins sub-battlefield (2, n) if $x_m > R_A - R_B$ and A wins it if $x_m < R_A - R_B$. B wins sub-battlefield (3, n) if $x_m > \bar{\bar{x}}_m$ and A wins it if $x_m < \bar{\bar{x}}_m$ where $\bar{\bar{x}}_m$ is defined by $h_{3n}(R_A - \bar{\bar{x}}_m) = h_{3n}(R_B) - \gamma_n$. B wins sub-battlefield (1, m) if $x_m < \bar{x}$, and A wins it if $x_m > \bar{x}$ where $\bar{x}$ is defined by $h_{1m}(\bar{x}) = \theta_m$. Given assumption (6), $R_A - R_B < \bar{x}$, while $R_A - R_B < \bar{\bar{x}}_m$ follows from the definition of $\bar{\bar{x}}$. From assumption (5), $R_B > \bar{x}$ while $R_A > \max [\bar{x}, \bar{\bar{x}}_m]$. Hence, there are three cases to consider: (a) $R_A - R_B < \bar{\bar{x}}_m < \bar{x} < R_B$, (b) $R_A - R_B < \bar{x} < \bar{\bar{x}}_m < R_B$, and (c) $R_A - R_B < \bar{x} < R_B < \bar{\bar{x}}_m$. In all three cases, if A's best reply is below R_B, then it is straightforward to show that for B to set y_m at $x_m + \varepsilon$ instead of at 0 raises B's expected payoff. In case (b), since $V^A = M$ for $x_m > \bar{\bar{x}}_m$ but equals $M + n_3$ for x_m between $\bar{x}$ and $\bar{\bar{x}}_m$, A's best reply is always below R_B. Then consider case (a) and $R_B \leq x_m < R_A$. If $m_1 > n_2 + n_3$, then such x_m are best replies by A to $y_m = 0$. Opponent B cannot set an alternative y_m at $x_m + \varepsilon$. Consider instead $y_m = R_B$. Then $V^B \geq m_1 + n_1$ using assumption (6) and concavity, whereas for $y_m = 0$, $V^B = N$ which is smaller given the parameter restriction. Hence, case

(a) cannot be an equilibrium. In case (c), $R_B \leq x_m < \bar{\bar{x}}_m$ is a best reply if $m_1 > n_2$. At this value, $V^B = n_1 + n_2$. If y_m is set at R_B instead of 0, $V^B = m_1 + n_1$, which is larger. Overall, no situation with $y_m = 0$ is a pure strategy equilibrium.

(iv) Consider $y_m = R_B$. As in (iii), if A's best reply is below R_B, then for B to choose $y_m = x_m + \varepsilon$ would raise B's payoff. If $R_B \leq x_m \leq R_A$ are best replies to $y_m = R_B$, then $x_n < R_B$ follows for $R_A < 2R_B$. Now set $y_n = x_n + \varepsilon$, which translates to $y_m = x_m + R_B - R_A - \varepsilon$. It is straightforward to calculate V^B which must exceed its value when $y_m = R_B$. Hence, this cannot be an equilibrium.

Second, assume $R_A = R_B \equiv R$. Consider $x_m = R$. The only sub-battlefields where the outcomes differ for different values of y_m are (3, m) and (1, n). The cutoffs are $h_{1m}(\bar{y}_m) = h_{1m}(R) - \theta_m$ and $h_{3n}(R - \bar{\bar{y}}_m) = \gamma_n$. For the different orderings of $\bar{y}_m$ and $\bar{\bar{y}}_m$, B's best reply can be calculated. For example, if $\bar{y}_m < \bar{\bar{y}}_m$, B's best reply is any y_m with $\bar{y}_m < y_m < \bar{\bar{y}}_m$ if $\frac{1}{2}n_2 + n_3 > \frac{1}{2}m_2$ or is $y_m = R$ if $\frac{1}{2}n_2 + n_3 < \frac{1}{2}m_2$. In the first case, for A to choose $x_m = y_m + \varepsilon$ instead of $x_m = R$ raises V^A while in the second, setting $x_m = 0$ raises V^A. Thus, for $\bar{y}_m < \bar{\bar{y}}_m$, $x_m = R$ cannot be an equilibrium. Similar arguments show $x_m = R$ cannot be an equilibrium when $\bar{y}_m \geq \bar{\bar{y}}_m$. Exactly symmetric arguments hold for $y_m = R$.

The only remaining possibilities for equilibria are when either $x_m = 0$ or $y_m = 0$. Whenever the best reply by B to $x_m = 0$ is some $y_m > 0$, this can be shown not to be an equilibrium. Thus, the only possibility is for both to equal 0. If both equal 0 and $n_2 < m_2$, then either could gain by a small increase in their strategy so $n_2 > m_2$ must hold in an equilibrium. When $x_m = y_m = 0$, $V^A = \frac{1}{2}m_2 + \frac{1}{2}n_2 + m_3 + n_3$ and $V^B = \frac{1}{2}m_2 + \frac{1}{2}n_2 + m_1 +$

n_1. When $\bar{y} < \bar{\bar{y}}$ (where these were defined prior to the statement of the theorem), the best interior values of y_m are $\bar{y} < y_m < \bar{\bar{y}}$ since B wins sub-battlefield (3, m) but does not lose the sub-battlefield (1, m). For such y_m, $V^B = M + n_1$. This is less than V^B when $y_m = 0$ if $\frac{1}{2}m_2 + m_3 < \frac{1}{2}n_2$. If $\bar{\bar{y}} \leq \bar{y}$, then the best reply by B to $x_m \neq 0$ can either be 0 or any value in one of the intervals $(0, \bar{\bar{y}})$ or $(\bar{y}, R)$. In this case, B's best reply is 0 instead of being in the other intervals if $\frac{1}{2}m_2 + m_3 < n_1 + n_2$. Thus, condition (A) is necessary and sufficient for $y_m = 0$ to be a best reply to $x_m = 0$. A similar argument shows that (B) is necessary and sufficient for $x_m = 0$ to be a best reply by A to $y_m = 0$.

Q.E.D.

Proof of Lemma 2: By definition, $h_m(\bar{y}) = \gamma_m$ and $h_n(R) - h_n(R - \bar{\bar{y}}) = \theta_n$, so $\gamma_m < \theta_n$ implies $h_m(\bar{y}) < h_n(R) - h_n(R - \bar{\bar{y}})$. By assumption, $h_n(\bar{y}) \leq h_m(\bar{y})$. Hence, if $\bar{y} \geq \bar{\bar{y}}$, a contradiction exists since $h_n(\bar{y}) \geq h_n(\bar{\bar{y}}) - h_n(0) \geq h_n(R) - h_n(R - \bar{\bar{y}})$ where the right inequality follows from the concavity of h. Hence, $\bar{y} < \bar{\bar{y}}$ must hold. If $\gamma_m = \theta_n$ and h_i is strictly concave, there is again a contradiction if $\bar{y} \geq \bar{\bar{y}}$ since $h_n(\bar{y}) \leq h_m(\bar{y}) = h_n(R) - h_n(R - \bar{\bar{y}})$ from the definitions of $\bar{y}$ and $\bar{\bar{y}}$, but $h_n(\bar{y}) \geq h_n(\bar{\bar{y}}) - h_n(0) > h_n(R) - h_n(R - \bar{\bar{y}})$ from strict concavity. An identical argument holds showing $\bar{x} < \bar{\bar{x}}$.

Q.E.D.

Proof of Theorem 2: For $x_m \in (R - \alpha, \alpha)$, given the definition of α, it follows that $x_m < \alpha(\gamma_m)$ and $R - x_n = x_m > R - \alpha(\gamma_m)$. Then for such x_m and for any y_m, A always wins sub-battlefields (3, j), j = m, n. Similarly, B always wins (1, i) regardless of y_i, since $x_i < \alpha \leq \alpha(\theta_i)$, i = m, n. Hence, for any $x_m \in (R - \alpha, \alpha)$, $EV^A(x_m, F) = m_3 + n_2 + n_3 + (m_2 - n_2)[F(x_m) - f^a(x_m)/2]$, since A also wins (2, m) with payoff m_2 if $y_m < x_m$, wins n_2 if $y_m > x_m$, and wins $(m_2 + n_2)/2$ if $y_m = x_m$. Similarly, for any $y_m \in (R - \alpha, \alpha)$, $EV^B(y_m, E) = m_1 + n_1 + n_2 + (m_2 - n_2)[E(y_m) - e^a(y_m)/2]$.

Now assume that A uses a mixed strategy that places positive probability on values of x_m in the open interval $(R - \alpha, \alpha)$. Let x_m^ℓ be the largest value of $x_m \leq \alpha$ for which $E(\alpha) - e^a(\alpha) > E(x_m^\ell - \varepsilon)$, all $\varepsilon > 0$, and let x_m^s be the smallest value of $x_m \geq R - \alpha$ such that $E(x_m^s + \varepsilon) > E(R - \alpha)$, all $\varepsilon > 0$. Thus, x_m^ℓ is the smallest value of x_m below α such that A places no probability weight between it and α while x_m^s is the largest value of x_m above $R - \alpha$ such that A places no probability weight between it and $R - \alpha$. The following seven steps show that F must then be constant on the interval.

(1) Assume there exists a $z \in (R - \alpha, \alpha)$ at which $e^a(z) > 0$. Then $f^a(z) = 0$ must hold. To see this, assume not and that $f^a(z) > 0$. Then, there exists $\varepsilon > 0$ such that $f^a(y_m) = 0$ for all $y_m \neq z$ and $z - \varepsilon < y_m < z + \varepsilon$. It follows that $F(z + \varepsilon) + f^a(z)/2 > F(z) = f^a(z) + \lim_{\delta \to 0} F(z - \delta) > f^a(z)/2 + F(z - \varepsilon)$. If $m_2 - n_2 > 0$, then $EV^A(z + \varepsilon, F) - EV^A(z, F) = (m_2 - n_2)[F(z + \varepsilon) - F(z) + f^a(z)/2] > 0$ and if $n_2 - m_2 > 0$ then $EV^A(z - \varepsilon, F) - EV^A(z, F) = (n_2 - m_2)[F(z) - F(z - \varepsilon) - f^a(z)/2] > 0$. In either case this contradicts $e^a(z) > 0$ since there exist values of x other than z that yield higher expected payoffs for A.

(2) Given the definition of x_m^ℓ one of the following must hold:

(i) $x_m^\ell < \alpha$ and $e^a(x_m^\ell) > 0$,

(ii) $x_m^\ell = \alpha$ or there exists a sequence $x_m^{\mathrm{k}} \rightarrow x_m^\ell$ with $x_m^{\mathrm{k}} < x_m^\ell$ and for which $\mathrm{e}^{\mathrm{a}}(x_m^{\mathrm{k}}) > 0$ for all k, or

(iii) $x_m^\ell = \alpha$ or $\mathrm{e}^{\mathrm{a}}(x_m^\ell) = 0$ and there exist intervals below and arbitrarily close to x_m^ℓ on which $E^*(x_m)$ is strictly increasing.

If (i), $\mathrm{f}^{\mathrm{a}}(x_m^\ell) = 0$ from the first point. Hence, $\mathrm{EV}^{\mathrm{A}}(x_m^\ell, \mathrm{F}) = m_3 + n_2 + n_3 + (m_2 - n_2)\mathrm{F}(x_m^\ell)$. If (ii), $\mathrm{f}^{\mathrm{a}}(x_m^{\mathrm{k}}) = 0$, all k, must hold with $\mathrm{EV}^{\mathrm{A}}(x_m^{\mathrm{k}}, \mathrm{F}) = m_3 + n_2 + n_3 + (m_2 - n_2)\mathrm{F}(x_m^{\mathrm{k}})$, all k. Since $x_m^{\mathrm{k}} \rightarrow x_m^\ell$, there must exist x_m' arbitrarily close to x_m^ℓ, at which $\mathrm{EV}^{\mathrm{A}}(x_m', \mathrm{F}) = m_3 + n_2 + n_3 + (m_2 - n_2)\mathrm{F}(x_m')$. In (iii), in any interval there exist points at which $\mathrm{f}^{\mathrm{a}}(y_m) = 0$. Hence, again there exist x_m' arbitrarily close to x_m^ℓ at which A places positive density and for which $\mathrm{EV}^{\mathrm{A}}(x_m', \mathrm{F}) = m_3 + n_2 + n_3 + (m_2 - n_2)\mathrm{F}(x_m')$.

(3) A similar argument shows that there exists an x_m'' arbitrarily close to x_m^s at which $\mathrm{EV}^{\mathrm{A}}(x_m'', \mathrm{F}) = m_3 + n_2 + n_3 + (m_2 - n_2)\mathrm{F}(x_m'')$ and at which either $\mathrm{e}^{\mathrm{a}}(x_m'') > 0$ or $E(x)$ is strictly increasing at x_m''.

(4) Since A is willing to place positive probability on both x_m' and x_m'', they must have the same expected payoff ($\mathrm{EV}^{\mathrm{A}}(x_m', \mathrm{F}) = \mathrm{EV}^{\mathrm{A}}(x_m'', \mathrm{F})$). This holds if and only if $\mathrm{F}(x_m') = \mathrm{F}(x_m'')$. Since x_m'' is arbitrarily close to x_m^s, $\mathrm{F}(x_m'') = \mathrm{F}(x_m^s)$ from right-hand continuity of F. Since x_m' is arbitrarily close to x_m^ℓ, then B places no probability in the (x_m^s, x_m^ℓ) interval. If $x_m^s = \mathrm{R} - \alpha$ and $x_m^\ell = \alpha$, the theorem follows immediately.

(5) Consider x_m^s > R – α and assume that F(x_m^s) > F(R – α). Then B either has an atom at x_m^s or places probability weight in the interval (R – α, x_m^s). If $n_2 - m_2 > 0$, take any x_m'' , $x_m^s \leq x_m'' \leq x_m^\ell$, at which A is willing to place positive probability and at which $f^a(x_m'') = 0$. Such x_m'' exist by the definitions of x_m^s and x_m^ℓ. Consider any x_m', $R - \alpha < x_m' < x_m^s$ at which $F(x_m'') \geq F(x_m^s) > F(x_m')$. Then $EV^A(x_m', F) - EV^A(x_m'', F) = (n_2 - m_2)[F(x_m'') - F(x_m') + f^a(x_m')/2] > 0$, contradicting A being willing to play x_m''. On the other hand, if $m_2 - n_2 > 0$, take $y_m' = x_m^s$ if $f^a(x_m^s) > 0$ or take any y_m' , $R - \alpha < y_m' < x_m^s$ at which B is willing to place positive probability if $f^a(x_m^s) = 0$. Then, for any y_m'' , $x_m^s < y_m'' \leq x_m^\ell$, $EV^B(y_m'', E) - EV^B(y_m', E) = (m_2 - n_2)[E(y_m'') - E(y_m') - e^a(y_m'')/2]$. Since $E(y_m'') = \lim_{\delta \to 0} E(y_m'' - \delta) + e^a(y_m'')$ and since $\lim_{\delta \to 0} E(y_m'' - \delta) > E(y_m') = E(R - \alpha)$ by definition of x_m^s, then $EV^B(y_m'', E) > EV^B(y_m', E)$, contradicting B being willing to play y_m'. Thus, regardless of the sign of $(n_2 - m_2)$, there is a contradiction of F(x_m^s) > F(R – α), so F(x_m^s) = F(R – α) must hold in any equilibrium.

(6) Finally, consider x_m^ℓ < α and assume that $F(\alpha) - f^a(\alpha) > F(x_m^\ell - \varepsilon)$, any ε > 0. If $m_2 - n_2 > 0$, consider any x_m'' , $x_m^s \leq x_m'' \leq x_m^\ell$ at which A is willing to place positive probability, and any x_m' , $x_m^\ell < x_m' < \alpha$, at which $F(x_m') > F(x_m'') = F(x_m^\ell - \varepsilon)$. Then $EV^A(x_m', F) - EV^A(x_m'', F) = (m_2 - n_2)[\lim_{\delta \to 0} F(x_m' - \delta) - F(x_m'') + f^a(x_m')/2] > 0$, contradicting A being willing to play x_m''. On the other hand, if m_2

$-$ n$_2$ < 0, take $y'_m = x^\ell_m$ if f^a(x^ℓ_m) > 0 or take any y'_m , $x^\ell_m < y'_m < \alpha$ at which B is willing to place positive probability if f^a(x^ℓ_m) = 0. For any y''_m , $x^s_m \le y''_m < x^\ell_m$, EV^B($y''_m$, E) - EV^B(y'_m , E) = (n$_2$ - m$_2$)[E(y'_m) - E(y''_m) + e^a(y''_m)/2] > 0, which contradicts B being willing to play y'_m . Regardless of the sign of (m$_2$ – n$_2$), there is a contradiction of F(α) - f^a(α) > F(x^ℓ_m - ε), so F(α) - f^a(α) = F(x^ℓ_m - ε) must hold.

(7) From steps (4), (5) and (6), F($\alpha - \varepsilon$) = F(x^ℓ_m - δ) = F(x^s_m) = F(R – α) for any small ε and δ. Hence B places no weight on the open interval (R – α, α).

Q.E.D.

Proof of Corollary 1: Assume there exists a mixed strategy equilibrium in which one of the players, say A, has positive probability of choosing an x$_m$, inside the interval (R – α, α), so that E(α) – e^a(α) > E (R – α). Denote this equilibrium strategy for A as S^A. Then, because the game is symmetric, there must exist another equilibrium with B having F(α) – f^a(α) > F (R – α). Call this strategy S^B. Since this is a two-person constant sum game, if a strategy for one player is ever part of an equilibrium, then it forms an equilibrium with any strategy ever an equilibrium for the other player. Hence, the pair (S^A, S^B) would form an equilibrium with both players being in the interval (R – α, α) with positive probability, contradicting Theorem 2. **Q.E.D.**

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Figure 1: Post-campaign advantages in each battlefield

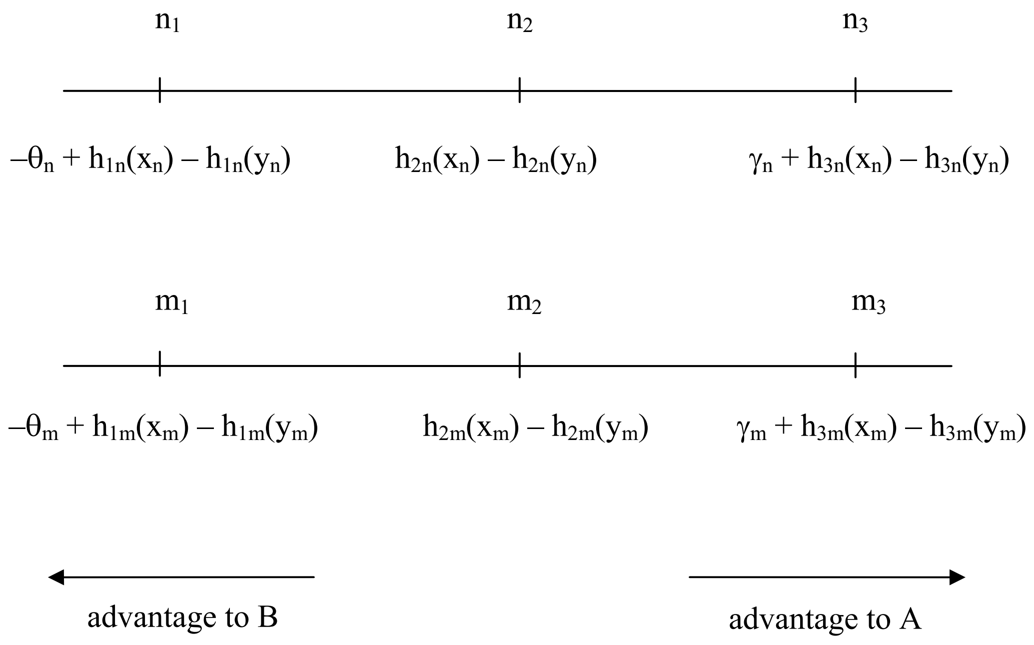


Table 1: Market size in states with only two markets or where one market is much larger than all other markets combined

State	Market[a]	Population (in hundred thousands)[b]	Fraction of state[c]	Fraction of market[d]
MD	Baltimore	27.2	0.54	1
	Washington, DC	23.6	0.46	0.45
NJ	New York	63.1	0.75	0.33
	Philadelphia	21.0	0.25	0.28
NM	Albuquerque	15.7	0.90	0.92
	El Paso	1.7	0.10	0.20
OR	Portland, OR	22.8	1.00	1
	Spokane	0.1	0.00	0.01
AZ	Albuquerque	0.7	0.01	0.04
	Phoenix	39.0	0.78	1
	Tucson	10.0	0.20	1
CO	Albuquerque	0.7	0.02	0.04
	Colorado Springs	7.9	0.19	1
	Denver	32.9	0.79	0.96
GA	Atlanta	51.0	0.82	0.99
	Chattanooga	2.5	0.04	0.30
	Greenville	0.9	0.01	0.05
	Jacksonville, FL	1.9	0.03	0.13
	Savannah	5.8	0.09	0.78
IL	Champaign	9.2	0.08	0.99
	Chicago	84.9	0.78	0.92
	Davenport	4.4	0.04	0.56
	Evansville	0.5	0.00	0.07
	Paducah	3.4	0.03	0.35
	St. Louis	5.9	0.05	0.24
NE	Denver	0.6	0.07	0.02
	Omaha	7.8	0.93	0.79
	Wichita	0.02	0.00	0.002
NH	Boston	9.2	0.82	0.16
	Burlington	1.2	0.11	0.15
	Portland, ME	0.8	0.07	0.08

VT	Albany, NY	0.4	0.07	0.03
	Boston	0.4	0.07	0.01
	Burlington	5.3	0.87	0.64

[a] Nielson market areas are made up of counties. Population data are the sums of the relevant county populations from the 2000 Census of Population and Housing.

[b] All market populations shown are the number of people in the market who live in the listed state. For example, the population shown for Washington, DC is the population of all counties in the DC broadcast market that fall within the state of Maryland.

[c] This is the population shown in the previous column divided by the total population of the state that lives within one of the 100 largest broadcast markets in the U.S. Since all of these markets are listed, these figures should sum to 1 for each state, with some allowance for rounding.

[d] This is the number from the "population" column divided by the total population of the market over all states. For example, the Washington, D.C. market covers part of the states of Maryland, Virginia and West Virginia, so the number in the "fraction of market" column would be the population of the D.C. market that is in Maryland as a fraction of the total population in the D.C. market in all states.

Table 2: Quantity of political advertising in states with only two markets, 2002

State	Office	Candidate	Market	Total ads	Fraction of total
MD	G	Ehrlich* (R)	Baltimore	1,817	0.40
		Ehrlich* (R)	Washington, DC	2,682	0.60
		Townsend (D)	Baltimore	1,990	0.49
		Townsend (D)	Washington, DC	2,086	0.51
NJ	S	Forrester (R)	New York	1,575	0.52
		Forrester (R)	Philadelphia	1,432	0.48
		Lautenberg* (D)	New York	1,355	0.56
		Lautenberg* (D)	Philadelphia	1,079	0.44
NM	S	**Domenici* (R)**	Albuquerque	1,755	1
		Domenici* (R)	El Paso	0	0
		Tristani (D)	Albuquerque	196	0.92
		Tristani (D)	El Paso	17	0.08
	G	Richardson* (D)	Albuquerque	2,523	0.73
		Richardson* (D)	El Paso	956	0.27
		Sanchez (R)	Albuquerque	1,876	0.85
		Sanchez (R)	El Paso	342	0.15
AZ	G	Napolitano* (D)	Phoenix	1,841	0.60
		Napolitano* (D)	Tucson	1,240	0.40
		Salmon (R)	Phoenix	2,136	0.65
		Salmon (R)	Tucson	1,126	0.35
CO	S	**Allard* (R)**	Colorado Springs	4,622	0.44
		Allard* (R)	Denver	5,765	0.56
		Strickland (D)	Colorado Springs	3,915	0.38
		Strickland (D)	Denver	6,293	0.62
	G	Heath (D)	Colorado Springs	72	0.29
		Heath (D)	Denver	178	0.71
		Owens* (R)	Colorado Springs	1,637	0.49
		Owens* (R)	Denver	1,687	0.51

Incumbents' names are shown in bold print. Winners are denoted with *, and the candidate's party is shown in parentheses after his or her name. Incumbency status, election outcomes and party are from CNN.com. Total ads are the number of advertising occurrences played for the candidate in the market, and include only those ads played after the primary. Fraction of total is the number from the "total ads" column divided by the total number of advertising occurrences for the candidate in all markets.

Table 3: Quantity of political advertising in states where one market is much larger than all other markets combined, 2002

State	Office	Candidate	Market	Total ads	Fraction of total
GA	S	Chambliss* (R)	Atlanta	2,783	0.50
		Chambliss* (R)	Chattanooga	1,081	0.20
		Chambliss* (R)	Savannah	1,674	0.30
		Cleland (D)	Atlanta	3,056	0.49
		Cleland (D)	Chattanooga	1,096	0.18
		Cleland (D)	Savannah	2,069	0.33
	G	**Barnes (D)**	Atlanta	4,504	0.49
		Barnes (D)	Chattanooga	1,127	0.12
		Barnes (D)	Savannah	3,566	0.39
		Perdue* (R)	Atlanta	604	0.58
		Perdue* (R)	Chattanooga	184	0.18
		Perdue* (R)	Savannah	254	0.24
IL	G	Blagojevich* (D)	Champaign	2,345	0.24
		Blagojevich* (D)	Chicago	3,283	0.34
		Blagojevich* (D)	Davenport	793	0.08
		Blagojevich* (D)	Paducah	2,304	0.24
		Blagojevich* (D)	St Louis	923	0.10
		Ryan (R)	Champaign	1,240	0.26
		Ryan (R)	Chicago	1,250	0.26
		Ryan (R)	Davenport	914	0.19
		Ryan (R)	Paducah	1,040	0.22
		Ryan (R)	St Louis	350	0.07
NH	S	Shaheen (D)	Boston	4,320	0.73
		Shaheen (D)	Burlington	347	0.06
		Shaheen (D)	Portland	1,222	0.21
		Sununu* (R)	Boston	3,509	0.89
		Sununu* (R)	Burlington	160	0.04
		Sununu* (R)	Portland	258	0.07

NH	G	Benson* (R)	Boston	1,032	1
		Benson* (R)	Burlington	0	0
		Benson* (R)	Portland	0	0
		Fernald (D)	Boston	245	1
		Fernald (D)	Burlington	0	0
		Fernald (D)	Portland	0	0

Incumbents' names are shown in bold print. Winners are denoted with *, and the candidate’s party is shown in parentheses after his or her name. Incumbency status, election outcomes and party are from CNN.com. Total ads are the number of advertising occurrences played for the candidate in the market, and only include ads played after the primary. Fraction of total is the number from the “total ads” column divided by the total number of advertising occurrences for the candidate in all markets.

Figure 2: Allocation of ads across market pairs

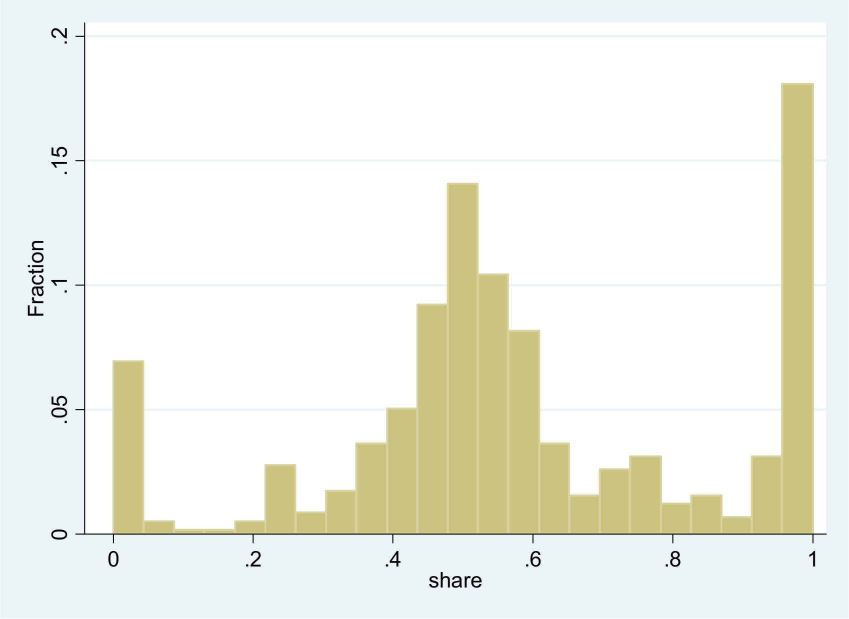
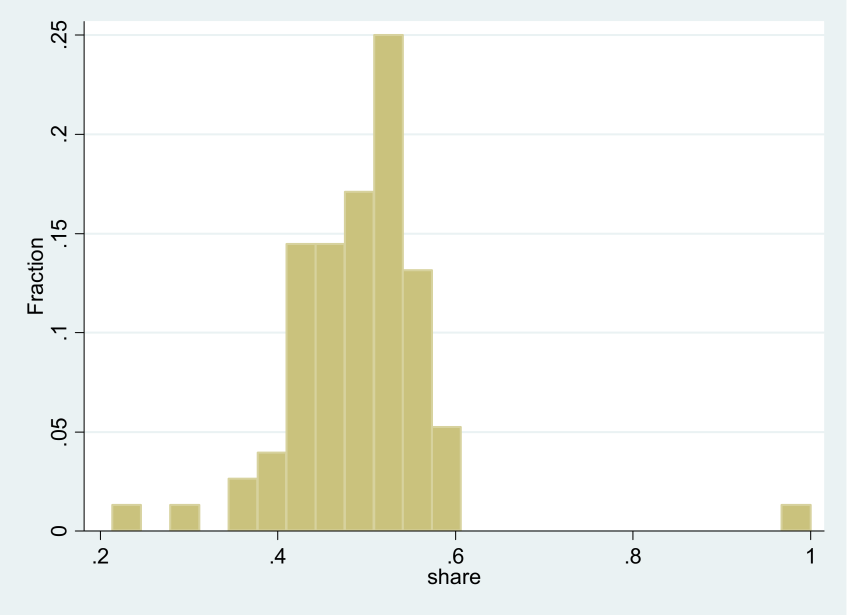
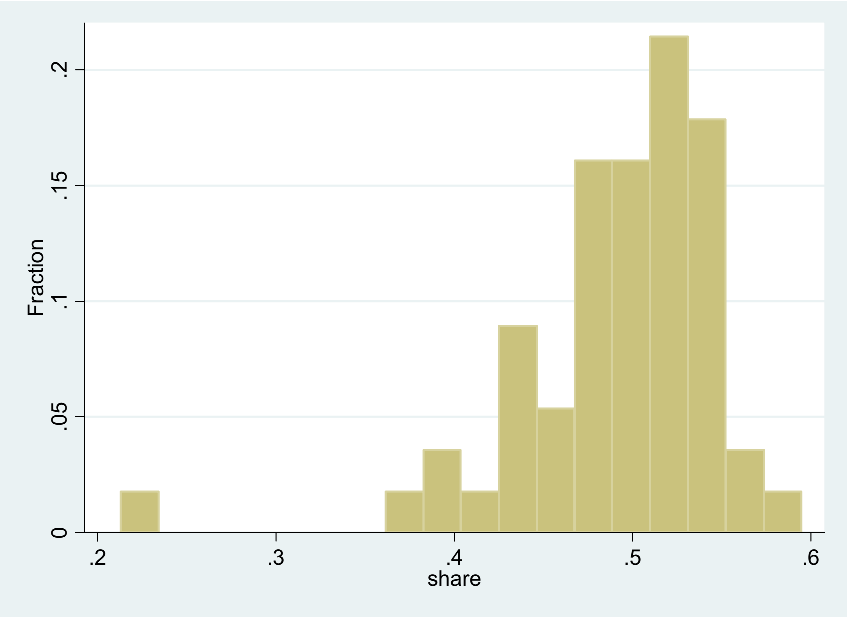


Figure 3: Allocation of ads across market pairs
when the candidate's opponent allocates symmetrically across the pair



Note: Symmetric allocation here is defined as allocating 48-52 or closer across the pair.

Figure 4: Allocation of ads across market pairs
when the candidate's opponent allocates symmetrically across the pair
and the candidates' resource constraints are not too different



Note: Symmetric allocation here is defined as allocating 48-52 or closer across the pair. "Resource constraints not too different" means that the candidate playing more total ads does not play a total (across all markets) of more than twice as many ads as his opponent.

Figure 5: Advertising allocations at different points in time