Quadratic Programming (QP)
Big Picture

1. QP is a special case of nonlinear programming

2. The objective function is quadratic, a second order polynomial of decision variables

3. Global minimum exists if the quadratic form is positive definite (or the function is strictly convex)

4. There may be constraints, which may or may not be binding
Example 1: Unconstrained QP

1. For example, consider minimizing a quadratic function without constraints

\[
\min -8x_1 - 16x_2 + x_1^2 + 4x_2^2
\]  

(1)

2. To see why this function has a minimum, we complete the square, and rewrite it as

\[
-8x_1 - 16x_2 + x_1^2 + 4x_2^2 = (x_1 - 4)^2 + 4(x_2 - 2)^2 - 32 \geq -32
\]  

(2)

where the last equality holds when \(x_1 = 4, x_2 = 2\). The global minimum value is -32.

3. Using jargon, there is a global minimum because the objective function is strictly convex.

4. Alternatively, we can solve this problem by taking (partial) derivatives, and setting them to zero

\[
\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0 \implies x_1 = 4, x_2 = 2
\]  

(3)

5. Next we show how R uses matrix form to solve QP problem
Inner Product and Quadratic Form I

From matrix algebra, we define inner product as a row vector multiplied by a column vector. Inner product is a scalar (a $1 \times 1$ matrix)

$$\begin{pmatrix} a \\ b \end{pmatrix} = ax_1 + bx_2$$  \hspace{1cm} (4)

A quadratic form is also a scalar, and it is a row vector multiplied by a squared matrix multiplied by the transpose of the row vector

$$\begin{pmatrix} x_1, x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$  \hspace{1cm} (5)

Exercise: use definition of inner product (4) to prove (5)
Inner Product and Quadratic Form II

1. We can use inner product to rewrite

\[-8x_1 - 16x_2 = -(8, 16) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} 8 \\ 16 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\]  

(6)

where \( A' \) is the transpose of \( A \). The transpose of a column vector is a row vector.

2. Moreover, we can use quadratic form to rewrite

\[x_1^2 + 4x_2^2 = (x_1, x_2) \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\]  

(7)

3. We organize our matrix in such a special way in order to use R package quadprog
**R Package “quadprog”**

The R package “quadprog” can solve the QP with the form of

$$\min -d^T b + \frac{1}{2} b^T D b \quad \text{(with constraint } A^T b \geq b_0)$$

where

1. $b$ is the column vector of decision variables
2. $D$ is the square matrix in the middle of quadratic form multiplied by 2
3. $d$ is the column vector specifying the linear part in the objective function
4. The constrain is given by $A^T b \geq b_0$

For Example 1, $b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$, $d = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$, $A = 0$, $b_0 = 0$. 
R codes

library(quadprog)
Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(0,0,0,0),2,2,byrow=T)
bvec = c(0,0)
solve.QP(Dmat,dvec,Amat,bvec)

`$solution`
[1] 4 2

$value
[1] -32

Note that Amat is a zero matrix, and bvec is a zero column vector. We get the same answer as using the calculus.
Example 2: a QP problem that has no minimum

Let’s change $D$ and see what happens

```r
Dmat = matrix(c(2,4,4,8),2,2,byrow=T)
solve.QP(Dmat,dvec,Amat,bvec)
```

Error in solve.QP(Dmat, dvec, Amat, bvec) :
matrix D in quadratic function is not positive definite!

Exercise:

1. write down the objective function
2. is there constraint?
3. take partial derivatives, and set them to zero. Can you solve them?
4. we have trouble here because $D$ is not positive definite!
Positive Definite Matrix–I

To investigate this issue, let’s complete the square for a general second-order polynomial

\[ ax_1^2 + bx_1 x_2 + cx_2^2 = a \left( x_1 + \frac{b}{2a} x_2 \right)^2 + \left( c - \frac{b^2}{4a} \right) x_2^2 \]  

(9)

It is positive for all \( x_1, x_2 \) (so a global minimum exists) only if

\[ a > 0, \text{ and } c - \frac{b^2}{4a} > 0 \]

(10)

Using quadratic form we have

\[ ax_1^2 + bx_1 x_2 + cx_2^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} a & b \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

(11)
Positive Definite Matrix–II

1. The matrix \[
\begin{pmatrix}
  a & b/2 \\
  b/2 & c
\end{pmatrix}
\]
is **positive definite** if and only if all its **leading principal minors** are strictly positive, i.e.,

\[ a > 0, \text{ and } ac - \frac{b^2}{2^2} > 0, \]

which is the same as (10). In short, there is global minimum only if the square matrix in the quadratic form is positive definite.

2. Exercise: show the square matrix is NOT positive definite for Example 2
Example 3: Constrained QP with non-binding constraints

1. Still consider
   \[ \min -8x_1 - 16x_2 + x_1^2 + 4x_2^2 \]

2. but now we add constraints
   \[ x_1 + x_2 \geq 5, x_1 \geq 3, x_1 \geq 0, x_2 \geq 0 \] (12)

3. Note that the unconstrained solution \( x_1 = 4, x_2 = 2 \) shown on page 7 satisfies (12). So constraint (12) has no effect, so is non-binding
Example 3: R

\[
D_{mat} = \text{matrix}(c(2,0,0,8),2,2,\text{byrow}=T)
\]
\[
d_{vec} = \text{c}(8,16)
\]
\[
A_{mat} = \text{matrix}(c(1,1,1,0),2,2,\text{byrow}=T)
\]
\[
b_{vec} = \text{c}(5,3)
\]
\[
\text{solve.QP}(D_{mat},d_{vec},A_{mat},b_{vec})
\]

\`
\`
solution
[1] 4 2

value
[1] -32
\`

Pay attention how we define Amat and bvec.
Amat and bvec

The way we define Amat and bvec is based on the matrix form of the constraint: The first two constraints in (12) can be rewritten as

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\geq
\begin{pmatrix}
5 \\
3
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}^T
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\geq
\begin{pmatrix}
5 \\
3
\end{pmatrix}
\]

Exercise: use definition of transpose matrix and inner product (4) to simplify the above inequality
Example 4: Some constraints are binding

Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(1,1,1,0),2,2,byrow=T)
bvec = c(5,4.5)
solve.QP(Dmat,dvec,Amat,bvec)

$`solution`
[1] 4.5 2.0
$value
[1] -31.75

Can you write down the constraints (Hint: look at Amat and bvec)? Here only \( x_1 \) changes to satisfy the new constraint. Note that the unconstrained minimum value -32 is less than the constrained minimum value -31.75. Does that make sense?
Example 5: Some constraints are binding

Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(1,1,1,0),2,2,byrow=T)
bvec = c(7,3)
solve.QP(Dmat,dvec,Amat,bvec)

`solution`
[1] 4.8 2.2
$value$
[1] -31.2

Can you write down the constraints? Here both $x_1$ and $x_2$ change to satisfy the new constraint.
Homework:

Still consider

$$\min \ -8x_1 - 16x_2 + x_1^2 + 4x_2^2$$

but the new constraint is

$$x_1 + x_2 \leq 5, x_1 \leq 3, x_1 \geq 0, x_2 \geq 0$$

(13)

Please use R to solve this QP problem. (Hint: $a < b$ is the same as $-a > -b$)